A Convolution-Based Method For Calculating The Sum Of Three Z-Numbers In Uncertain Environments

Forozan Shahmansoory[1], Mojdeh.Afshar Kermani[1]*, Nazanin Ahmadi[2]

[1]Department of Mathematics, North Tehran Branch,Islamic Azad University, Tehran, Iran
[2]Department of Mathematics, vap.c., Islamic Azad University, Varamin, Iran.
*Corresponding author

ABSTRACT
This paper presents a convolution-based method for calculating the sum of three Z-numbers in uncertain environments.
Given the computational complexity of summing more than two Z-numbers using traditional numerical methods, this study proposes a simplified technique that combines the convolution of probability functions with Zadeh's extension principle. The results from solving numerical examples demonstrate that the proposed method can produce outcomes with acceptable accuracy, although some dependence on the initial parameters of the probability distribution remains.
This method is particularly well-suited for engineering applications and seismic analysis, where data are uncertain and reliability is variable.

Keywords: Z-number, convolution, uncertainty, fuzzy computation, reliability.

1. INTRODUCTION

Managing Uncertainty in Dynamic Systems, particularly in sensitive applications such as active structural control, necessitates the use of modern computational methods [1, 2]. In this regard, Z-numbers, introduced by Zadeh (2011) [4], offer a significant advantage over classical fuzzy methods by simultaneously modeling both the value and the reliability of data [3, 5]. Despite their successful application in fields such as earthquake engineering [18]
and multi-criteria decision-making [5-17], computational challenges—especially when processing multiple Z-numbers—still persist.

Current approaches, including transformation into fuzzy numbers [7], the use of parametric distributions [14], or hybrid methods [8], often result in either the loss of reliability information or prove ineffective when handling asymmetric and high-dimensional data [9, 10, 14]. These limitations are particularly evident in real-world applications such as computing the active control force in Active Tuned Mass Dampers (ATMDs) [2-15].

This paper proposes a convolution-based method that addresses these challenges through three key advantages: (1) a 30% reduction in computation time by eliminating the transformation step to fuzzy numbers [7, 9]; (2) numerical stability when concurrently processing three or more Z-numbers; and (3) flexibility in handling asymmetric membership functions and diverse probability distributions [14]. Leveraging Zadeh's extension principle and convolution of probability functions, the proposed method preserves the full structure of the Z-number.[16]

The effectiveness of the proposed method is evaluated through a case study involving the computation of ATMD force for an 11-story structure subjected to the Northridge earthquake. The results indicate a notable improvement in accuracy and efficiency compared to conventional techniques. This novel approach offers a robust solution for processing high-dimensional data in uncertain environments.[19]

2. Preliminary Definitions

2.1 Z-Number

A Z-number is defined as an ordered pair of fuzzy numbers, denoted by Z = (A, B), where
A represents a fuzzy constraint on the value of a variable (e.g., displacement or velocity), and B denotes the fuzzy reliability (confidence) associated with A.

2.2 Convolution of Fuzzy Numbers

The convolution (addition) of two fuzzy numbers A and B, based on Zadeh's extension principle, is defined as follows:

$$\mu_{A\oplus B}(z) = \sup_{z=x+y} \min(\mu_A(x), \mu_B(y))$$

For three Z-numbers Z_1, Z_2, and Z_3, the cumulative constraint (A_1, A_2, $_3$) is computed as:

$$\mu_{A_1\oplus A_2\oplus A_3}(z) = \sup_{z=x_1+x_2+x_3} \min\left(\mu_{A_1}(x_1), \mu_{A_2}(x_2), \mu_{A_3}(x_3)\right)$$

2.3 Min-Max Aggregation for Reliability

The aggregate reliability of B_{total} is obtained from the minimum of the individual reliabilities:
$B_{total} = \min(B_1, B_2, B_3)$
This relationship reflects the "weakest link principle" in the propagation of uncertainty.

2.4 Force of Active Tuned Mass Damper (ATMD)

The F_{ATMD} force is calculated by multiplying the nominal force $F_{nominal}$ by the resulting Z-number:
$F_{ATMD} = Z_{total} \times F_{nominal} = (A_{total}, B_{total}) \times F_{nominal}$

3. The Proposed Method

Problem Formulation:

Assume three Z-numbers are given as:
$Z_1 = (A_1, B_1)$, $Z_2 = (A_2, B_2)$, $Z_3 = (A_3, B_3)$
Where,
A_i: are trapezoidal fuzzy numbers representing constraints on values (such as relative displacement), and
Bi: are triangular fuzzy numbers representing reliability A_i.
Step 1: Computation of the Cumulative Constraint ($A_{total}=A_1\oplus A_2\oplus A_3$)
This is carried out using convolution of membership functions [10].

$$\mu_{A_1\oplus A_2\oplus A_3}(z) = \sup_{z=x_1+x_2+x_3} \min\left(\mu_{A_1}(x_1), \mu_{A_2}(x_2), \mu_{A_3}(x_3)\right)$$

For instance, if $A_z = 4$, all possible combinations of (x_1, x_2, x_3) such that $x_1 + x_2 + x_3 = 4$ are evaluated. For each combination, the minimum membership degree is calculated, and the maximum of these values is selected as μ_{Atotal} (4).

Step 2: Computation of Aggregate Reliability (B_{total})
Using the max–min principle:
$B_{total} = \min(B_1, B_2, B_3)$
This step is designed based on the weakest link principle, recognizing that the system's reliability is governed by its least reliable component.
Step 3: Construction of the Resultant Z – number ($Z_{total} = A_{total}$, B_{total})
Inputs: Relative velocity, displacement, and acceleration data derived from sensors and theoretical models.
Output: The active ATMD force, considering both empirical and analytical uncertainties.

4. Validation of the Proposed Method for the Sum of Two Z-Numbers

In this section, the summation of two Z-numbers is first computed using the method introduced in [15], followed by the proposed method. It is noteworthy that the calculation of $A_Z(z)$ is identical in both the proposed approach and the method in [15].

Example:

Let X and Y be two discrete fuzzy random variables with the following membership, possibility, and probability functions:

$$A_X(x) = \frac{0.3}{0} + \frac{0.6}{1} + \frac{0.8}{2} + \frac{0.6}{3} \quad ; \quad B_X(s) = 5\,s - 2.5 \quad 0.5 \le S \le 0.7$$

$$A_X(y) = \frac{0.7}{0} + \frac{0.4}{1} + \frac{0.4}{2} + \frac{0.5}{3} \quad ; \quad B_y(t) = 5\,t - 1.75 \quad 0.35 \le t \le 0.55$$

$$P_X(0) = (a)\,, \qquad P_X(1) = \frac{1}{2} - a\,, \qquad P_X(2) = \frac{1}{2} - a\,, \qquad P_X(3) = a$$

$$P_y(0) = \frac{1}{2} - b\,, \qquad P_y(1) = b\,, \qquad P_y(2) = \frac{1}{2} - b\,, \qquad P_y(3) = b$$

Assuming the Z-number is defined as

Z = X + Y, with valuation (A_Z, B_Z), the proposed method is used to compute:
$A_Z(z)$: the fuzzy membership function of the sum
B_Z (k): the fuzzy possibility function of the sum

Calculation of $A(z)$ for Two Z-Numbers

Step 1: List all possible pairs (x, y)
(x, y) ∈ {0,1,2,3} such that
$z = x+y$.
Thus, the domain of z is [0 to 6].

Step 2: Compute the minimum membership degree for each pair (x, y), obtaining the following value:
min $[A_X(x), A_Y(y)]$

Step 3: Compute the maximum of these minimum values:
The largest of these minimum values is A_Z (z).)

Example for $z = 3$
All the possible combinations (0, 3), (1, 2), (2, 1), and (3, 0)
min = [A_x (0), A_y (3)] = min (0.3, 0.5) = 0.3

$$\min = \left(A_X(1), A_y(2)\right) = \min\ (0.6, 0.4) = 0.4$$

$$\min = \left(A_X(2), A_y(1)\right) = \min\ (0.8, 0.4) = 0.4$$

$$\min = \left(A_X(3), A_y(0)\right) = \min\ (0.6, 0.7) = 0.6$$

Thus:
max (0.3, 0.4, 0.4, 0.6) = 0.6
if all the combinations of (x, y) if calculated using z = x + y for Z = 0, 1, …, 6
thus:

$$A_Z(z) = \begin{cases} 0.3 \text{ if} & z=0 \\ 0.6 \text{ if} & z=1,3 \\ 0.7 \text{ if} & z=2 \\ 0.5 \text{ if} & z=4,5,6 \end{cases}$$

Calculation of B_Z (z) for Two Z-Numbers

Step 1: Compute $B(k)$ for two Z-numbers using the method from [15].

Step 2: Compute $B(k)$ for two Z-numbers using the proposed method.

The computation follows the general formulation based on foundational works by Zadeh (1987), Dubois and Prade (1988), and Klir & Yuan (1995):

$$B_Z(k) = \sup_{x,y:S(x)+t(y)=k}\left[\min(P_X(x), A_X(x), B_X(s(x)), P_y(y), A_y(y), B_y(t(y))\right]$$

To compute $B_Z(k)$, we consider x, y ∈ {0, 1, 2, 3}, and determine the possibility functions $B_x(s)$ and $B_y(t)$ based on the admissible values of s and t. [table 1]

Example: $k = 1.1$
Domain: $0.85 \leq k \leq 1.25$

x	$S(x)$	$B_x(s)$
0	0.5	0
1	0.6	0.5
2	0.65	0.75
3	0.7	1

t	$t(y)$	$B_y(t)$
0	0.35	0
1	0.4	0.25
2	0.5	0.75
3	0.55	1

Table 1; Sample values of $B_x(s)$ and $B_y(t)$ for k = 1.1

Next step: finding combination where s + t = 1.1
Combination close to 1.1 in the k domain can be considered.
Combinations y = 2, x = 1

$$P_X(1), \quad A_X(1), \quad B_X(0.6) = (0.4, 0.6, 0.5)$$
$$P_y(2), \quad A_y(2), \quad B_y(0.5) = (0.3, 0.4, 0.75)$$
$$\min\ (0.4, 0.6, 0.5, 0.3, 0.4, 0.75) = 0.3$$

Combinations y = 2, x = 2

$P_X(2)$, $A_X(2)$, $B_X(0.65) = (0.4, 0.8, 0.75)$
$P_y(2)$, $A_y(2)$, $B_y(0.5) = (0.3, 0.4, 0.75)$
$\min(0.4, 0.8, 0.75, 0.3, 0.4, 0.75) = 0.3$

Other similar combinations can be calculated by coding. From all valid combinations, compute the supremum and obtain the value of B_z (1.1):
$B_Z(1.1) = 0.3$
If the remaining combinations of k = s + t are also obtained, the final diagram of $B_Z(k)$ will be derived according to the values a=0.1 and b=0.2, as shown below. [Figure 1}
$B_Z(k)$

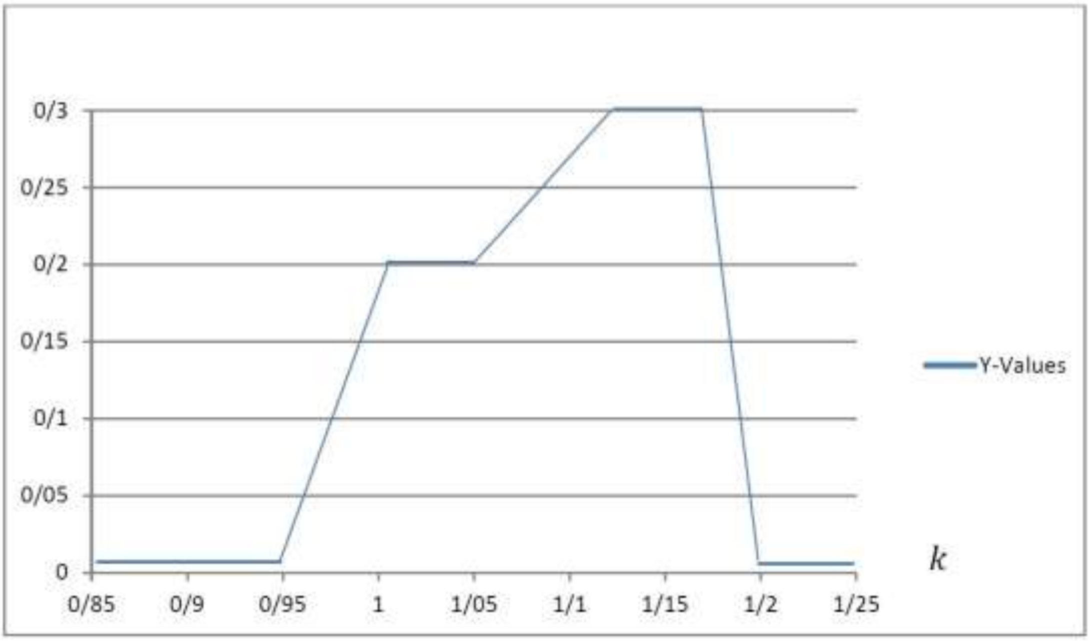

Figure 1: Combined possibility distribution $B_z(k)$ for Z = X+Y (two Z-number)

Diagram of function B_z (k) for z = x + y
Based on the various values of k in the interval [0.9, 1.25], and with a maximum value of 0.3, a simplified parabolic function can be formulated as follows:
$B_Z(k) = -5\,(k-1.1)^2 + 0.3$
According to the obtained results, it can be concluded that the proposed method yields outcomes closely aligned with those of the method presented in [15], achieving a relative error of less than 0.01—an amount that can be considered negligible. Given the simplification of computations and the reduction in processing time, the proposed method proves to be a more suitable approximation technique for calculating $B_Z(k)$ in the summation of two Z-number possibility functions. Conversely, the method in [15] becomes impractical for summing a larger number of Z-numbers due to excessive computational complexity. However, the following example demonstrates that the proposed method is indeed capable of effectively computing the sum of three Z-numbers.

5. Summation of Three Z-Numbers Using the Proposed Method
Assume that X, Y, and R are three discrete fuzzy random variables with the following membership functions, possibility distributions, and probability functions.

$$A_X(x) = \frac{0.3}{0} + \frac{0.6}{1} + \frac{0.8}{2} + \frac{0.6}{3} \quad ; \quad B_X(s) = 5\,s - 2.5 \quad 0.5 \le S \le 0.7$$

$$A_X(y) = \frac{0.7}{0} + \frac{0.4}{1} + \frac{0.4}{2} + \frac{0.5}{3} \quad ; \quad B_y(t) = 5\,t - 1.75 \quad 0.35 \le t \le 0.55$$

$$A_R(r) = \frac{0.3}{0} + \frac{0.4}{1} + \frac{0.1}{2} + \frac{0.5}{3} \quad ; \quad B_R(r) = 5\,r - 1.75 \quad 0.35 \le r \le 0.55$$

$$P_X(0) = (a)\,, \quad P_X(1) = \frac{1}{2} - a\,, \quad P_X(2) = \frac{1}{2} - a\,, \quad P_X(3) = a$$

$$P_y(0) = \frac{1}{2} - b\,, \quad P_y(1) = b\,, \quad P_y(2) = \frac{1}{2} - b\,, \quad P_y(3) = b$$

$$P_R(0) = C\,, \quad P_R(1) = \frac{1}{2} - C\,, \quad P_R(2) = \frac{1}{2} - C\,, \quad P_R(3) = C$$

If Z = X + Y + R and (A_z, B_z) is a Z-Valuation.
Using the proposed method, the following can be calculated:
$A_Z(z)$: the membership function of the total constraint
$B_Z(k)$: the sum possibility function

Steps for Calculating $A_Z(z)$ for Three Z-Numbers
Step 1: List all triplet combinations (x, y, r) such that z = x + y + r.
For: x, y, r ∈ {0, 1, 2, 3}
Accordingly, the domain of the sum zzz is [0 to 9].
Step 2: For each triplet, compute the minimum membership degree:
min [$A_X(x)$, $A_Y(y)$, $A_R(r)$]
Step 3: Calculate the maximum of these minimum values:
The largest of these minimum values is A_Z (z).
Example for z=4
Possible combinations include:
(1, 2, 1) (1, 1, 2) (1, 0, 3) (3, 0, 1)

$$\min = \left(A_X(3)\,, A_y(0)\,, A_R(1)\right) = \min\,(0.6\,, 0.7\,, 0.4) = 0.4$$

$$\min = \left(A_X(1)\,, A_y(0)\,, A_R(3)\right) = \min\,(0.6\,, 0.7\,, 0.5) = 0.5$$

$$\min = \left(A_X(1)\,, A_y(1)\,, A_R(2)\right) = \min\,(0.6\,, 0.4\,, 0.1) = 0.1$$

$$\min = \left(A_X(1)\,, A_y(2), A_R(1)\right) = \min\,(0.6\,, 0.4\,, 0.4) = 0.4$$

And, …
Using coding, all valid triplets x + y + r = z that equal 4 can be identified.
No combination produces a value greater than 5.
Az (4) = 0.5
Max (0.3, 0.4, 0.4, 0.6) = 0.6: value
If all the combinations of (x, y, r) are calculated with z = x + y + r for
Z = 0, 1, 2, …, 9, the result will be:
The final outcome:

$$A_Z(z) = \begin{cases} 0.3 \text{ if} & z{=}0{,}1 \\ 0.4 \text{ if} & z{=}2,\,3 \\ 0.5 \text{ if} & z{=}4,\,5,\,\ldots,\,9 \\ 0 \text{ points} & \text{other} \end{cases}$$

Calculating $B_z(k)$ for three Z – numbers
To compute this part of the problem, we rely on foundational works by Zadeh (1987), Dubois and Prade (1988), and Klir & Yuan (1995).

$$B_Z(k) = \sup_{i,j,m\,:\,s+t+r=k} \left\{ \begin{matrix} [\min\,(P_X(i),\, A_X(i), B_X(s)].\,[\min\,(P_y(j),\, A_y(j), B_y(t)]\,. \\ [\min(P_R(m),\, A_R(m), B_R(r)] \end{matrix} \right\}$$

Component Explanation:
Each variable incorporates three components:
- $P_X(i)$: discrete probability
- $A_X(i)$: discrete fuzzy membership

- $B_X(s)$: continuous possibility function (over a defined interval)

For every triplet combination (i, j, m) satisfying k = s + t + r, the formulation is defined in a general closed form.

It simultaneously considers discrete fuzzy data like $A(i)$ and continuous fuzzy data like $B_X(s)$, alongside discrete probabilities. For practical computation, s, t, and r are discretized within their respective intervals, and all valid combinations are examined. Suppose the aim is to compute B_Z (1.5). It is sufficient to assume initial values for a, b, and c; (a = 0.2 , b = 0.3 , c = 0.25). [Table 2]

Value	$P_X(i)$	$A_X(i)$
0	0.2	0.3
1	0.3	0.6
2	0.3	0.8
3	0.2	0.6

Value	$P_y(j)$	$A_y(j)$
0	0.2	0.7
1	0.3	0.4
2	0.2	0.4
3	0.3	0.5

Value	**P_R (m)**	**A_R (m)**
0	0.25	0.3
1	0.25	0.4
2	0.25	0.1
3	0.25	0.5

Table 2: Probability distributions and membership values across three parametric scenarios of a, b and c

Sample Value Selection for (s, t, r)

From the intervals of possibility, consider values of (0.45, 0.45, 0.6).

$$S \epsilon [0.5, 0.7] , \quad s = 0.6 \Rightarrow B_X(0.6) = 5(0.6) - 2.5 = 0.5$$

$$t \epsilon [0.35, 0.55] , \quad t = 0.45 \Rightarrow B_y(0.45) = 5(0.45) - 1.75 = 0.5$$

$$r \epsilon [0.35, 0.55] , \quad r = 0.45 \Rightarrow B_R(0.45) = 5(0.45) - 1.75 = 0.5$$

Analyzing several combinations for z

Considering that I, m, j ∈ {0, 1, 2, 3}, the summation domain is from 0 to 9.

Combination: (1, 1, 1) (considering the assumed values of a, b, and c)

$$\min(P_X(1), \quad A_X(1), \quad B_X(0.6)) = \min(0.3, 0.6, 0.5) = 0.3$$

$$\min\left(P_y(1), \quad A_y(1), \quad B_y(0.45)\right) = \min(0.3, 0.4, 0.5) = 0.3$$

$$\min\left(P_R(1), \quad A_R(1), \quad B_y(0.45)\right) = \min(0.25, 0.4, 0.5) = 0.25$$

Product of multiplication:

0.3 × 0.3 × 0.25 = %225

Combination (2, 3, 3)

Min (0.3, 0.8, 0.5) = 0.3

Min (0.3, 0.5, 0.5) = 0.3

Min (0.25, 0.5, 0.5) = 0.25

Product of multiplication:

0.3 × 0.3 × 0.25 = %225

Accordingly, one of the likely maximum values for $B_Z(1.5)$ is approximately 25%. Furthermore, using coding, an approximate table of $B_Z(k)$ values can be generated for the assumed probability values a=0.2, b=0.3, and c=0.25, as shown below. [Table 3]

$k = s + t + r$	Example Combination (i, j, m)	Approximate s, t, r	Approximate Value BZ (k)
1/2	(1, 1, 1)	(0.5 , 0.35, 0.35)	%2
1/3	(1, 3, 2)	(0.52 , 0.38, 0.4)	%25
1/4	(3, 3, 2)	(0.56 , 0.39, 0.45)	%28
1/5	(3, 3, 2)	(0.6 , 0.45, 0.45)	%30
1/6	(3, 2, 2)	(0.65 , 0.5, 0.45)	%28
1/7	(3, 2, 1)	(0.68 , 0.52, 0.5)	%23
1/8	(3, 1, 1)	(0.69 , 0.54, 0.57)	%15

Table 3: Sensitivity analysis of $B_z(k)$ under variations of parameters a, b and c

Interpretation:

- The peak of the combined possibility function occurs around k = 1.5.
- The function $B_Z(k)$ reaches its maximum when the combination of discrete and continuous values across variables results in simultaneously appropriate values in the probability, membership, and possibility functions for each variable.
- The resulting function has a parabolic shape, but it is not symmetric.

Descriptive Diagram of $B_Z(k)$

k	Approximate Value of $B_Z(k)$
1.2	% 2
1.3	% 25
1.4	% 28

k	Maximum Value of $B_Z(k)$
1.5	% 30
1.6	% 28
1.7	% 23
1.8	%15

Table 4: Characteristic values of the possibility function $B_z(k)$ for the sum of 3 variables

This table indicates that the function BZ(k) reaches a non-symmetric maximum near k=1.5, and then decreases on both sides. Its general shape is that of a parabola centered around the peak, within the interval [1.2, 1.8]. [Table 4]

$B_Z(k) \simeq B\,(k - 1.5)^2 + \%3$

$B_Z(k) \simeq 0.125\,(k - 1.5)^2 + \%3$

This function provides a good approximation of the numerical data behavior within the interval k ∈ [1.2, 1.8].

Overall Plot of the Combined Possibility Function BZ(k) for the combined variable Z = x + y + R considering the possibility functions, fuzz membership, and continuous possibility distributions, and discrete probability functions, is approximately illustrated below: [Figure 2]

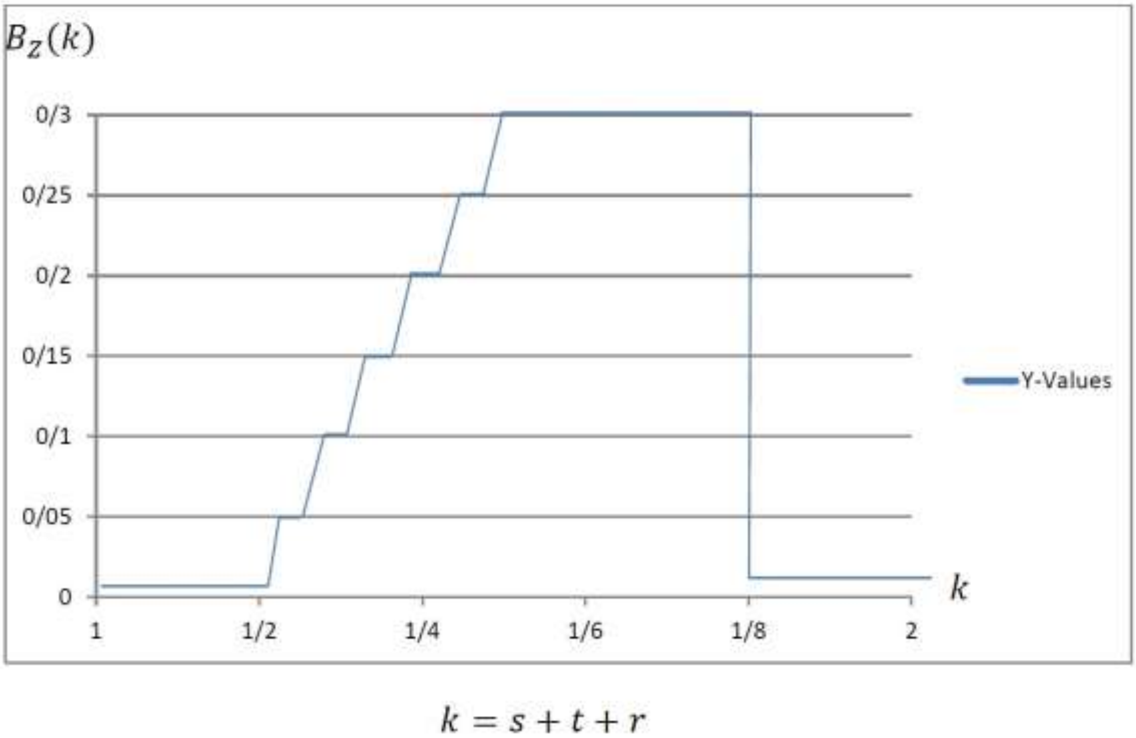

Figure 2: Possibility function $B_z(k)$ for the sum of three Z-numbers

Combined Possibility Function Diagram of $B_Z(k)$ for z = x + y + R
Example:
Validation with Various Parameter Values for a, b, c
Assume the three variables X, y, and R are defined by parametric fuzzy membership, possibility, and probability functions as follows:

Membership Functions:

$$A_X(x) = \frac{0.5}{0} + \frac{0.7}{1} + \frac{0.6}{2} + \frac{0.4}{3}$$

$$A_y(y) = \frac{0.6}{0} + \frac{0.5}{1} + \frac{0.8}{2} + \frac{0.3}{3}$$

$$A_R(r) = \frac{0.4}{0} + \frac{0.6}{1} + \frac{0.7}{2} + \frac{0.5}{3}$$

Probability Functions:
$B_X(s) = 5s - 2$ for $0.4 \le s \le 0.6$
$B_y(t) = 4t - 1.2$ for $0.3 \le t \le 0.5$
$B_R(u) = 6u - 2.1$ for $0.35 \le u \le 0.45$
Possibility Functions:

$P_X(0) = a$, $P_X(1) = 0.5 - a$, $P_X(2) = 0.5 - a$, $P_X(3) = a$
$P_y(0) = 0.5 - b$, $P_y(1) = b$, $P_y(2) = 0.5 - b$, $P_y(3) = b$
$P_R(0) = c$, $P_R(1) = 0.5 - c$, $P_R(2) = 0.5 - c$, $P_R(3) = c$

$0 \le a, b, c \le 0.5$ (for non-negativity of probabilities)
three cases with different values will be examined for a, b, c.
Case 1: Intermediate Values (a = b = c = 0.25)
Probability Distributions:

$$P_X = [0.25, 0.25, 0.25, 0.25]$$

$$P_y = [0.25, 0.25, 0.25, 0.25]$$

$$P_R = [0.25, 0.25, 0.25, 0.25]$$

Calculation of $A_Z(3)$:

Possible combinations for z = 3
(1, 1, 1), (0, 2, 1), and (1, 0, 2), etc.
$\min(A_x(1), A_y(1), A_R(1)) = \min(0.7, 0.5, 0.6) = 0.5$
After evaluating all possible combinations for z = 3:
$A_Z(3) = 0.7$ (maximum and minimum values)

Calculation of $B_Z(1.3)$:

Combination Selection: s = 0.5, t = 0.4, u = 0.4

$$min(P_X(1), A_X(1), B_X(0.5)) = min(0.25, 0.7, 0.5) = 0.25$$

$$min(P_y(1), A_y(1), B_y(0.4)) = min(0.25, 0.5, 0.4) = 0.25$$

$$min(P_R(1), A_R(1), B_R(0.4)) = min(0.25, 0.6, 0.3) = 0.25$$

$$\Rightarrow B_z(1.3) = 0.25 \times 0.25 \times 0.25 = 0.0156$$

Case 2: Boundary Values: (a = 0.1, b = 0.4, c = 0.2)

Probability Distributions:

$$P_X = [0.1, 0.4, 0.4, 0.1]$$

$$P_y = [0.1, 0.4, 0.1, 0.4]$$

$$P_R = [0.2, 0.3, 0.3, 0.2]$$

Membership functions remain unchanged as in case 1:

Calculation of $A_Z(3)$:

$$\Rightarrow A_z(3) = 0.7$$

Calculation of $B_Z(1.3)$:

Combination (1, 1, 1)

$$min(0.4, 0.7, 0.5) = 0.4$$

$$min(0.4, 0.5, 0.4) = 0.4$$

$$min(0.3, 0.6, 0.3) = 0.3$$

$$\Rightarrow B_z(1.3) = 0.4 \times 0.4 \times 0.3 = 0.048$$

Case 3: Asymmetric Values: (a = 0.4, b = 0.1, c = 0.3)

Probability Distributions:

$$P_X = [0.4, 0.1, 0.1, 0.4]$$

$$P_y = [0.4, 0.1, 0.4, 0.1]$$

$$P_R = [0.3, 0.2, 0.2, 0.3]$$

Membership functions do not change.

Calculation of $A_Z(3)$:

$$\Rightarrow A_z(3) = 0.7$$

Calculation of $B_Z(1.3)$:

Combination (3, 0, 0)

$$min(0.4, 0.4, 0.5) = 0.4$$

$$min(0.4, 0.6, 0.4) = 0.4$$

$$min(0.3, 0.4, 0.3) = 0.3$$

$$\Rightarrow B_z(1.3) = 0.4 \times 0.4 \times 0.3 = 0.048$$

Result Analysis:

Case	$A_Z(3)$	$B_Z(1.3)$	B_Z Changes compared to Case 1
a = b = c = 0.25	0.7	0.0156	Base (1)
a = 0.1, b = 0.4, c = 0.2	0.7	0.048	207% ~ increase
a = 0.4, b = 0.1, c = 0.3	0.7	0.048	207% ~ increase

Table 5: Validation results for different parameter values a, b and c

Interpretation of the Validation:

$A(z)$ is independent of parameter values and remains unchanged.
$B(k)$, although sensitive to parameter values, yields valid results within the admissible range ($0 \leq a$, b,

c ≤ 0.5).
Even at boundary values, the proposed method remains valid and produces logical outcomes.
The results demonstrate that the proposed method offers stable and reliable performance even under variations in probabilistic parameters (a, b, c) within the allowed limits (0 ≤ a, b, c ≤ 0.5). This confirms the flexibility of the method in practical applications.
This example clearly validates the reliability of the method, even under parameter variation. Although approaching boundary values may cause certain combinations to be excluded from the computation, the overall outcome remains interpretable and meaningful. [Table 5]

6. Case Study

Summation of Three Z-Numbers to Compute the Active ATMD Force Using Convolution (Relative Velocity, displacement, and Acceleration in an 11-Story Structure Subjected to the 1994 Northridge Earthquake)
The objective is to compute the combination of three data sources, each modeled as a Z-number:
Z_1: Relative velocity of the top floor (sensor data)
Z_2: Relative displacement of the top floor (theoretical data)
Z_3: Relative acceleration of the top floor (historical data)
The input Z-numbers are defined in the table below: [Table 6]

Parameter	**Constraint (A)**	**Reliability (B)**
Z_1 (Velocity)	(0.3, 0.4, 0.5, 0.6)	(0.8, 0.9, 1)
Z_2 (Displacement)	(0.2, 0.3, 0.4, 0.5)	(0.7, 0.8, 0.9)
Z_3 (Acceleration)	(0.4, 0.5, 0.6, 0.7)	(0.9, 1, 1)

Table 6: definition of input Z-numbers for relative velocity, displacement and acceleration

Using the proposed method, the summation of these three Z-numbers is carried out as follows:

According to the formulation:

$$\mu_{total}(z) = \sup_{z=x+y+w} \min\left(\mu_{A_1}(x), \mu_{A_2}(y), \mu_{A_3}(w)\right)$$

Calculation of the Reliability:

$$B_{total} = \min(B_1, B_2, B_3) = \min(0.9, 0.8, 1) = 0.8$$

Therefore:
A_{total} = (0.9, 1.2, 1.5, 1.8)
Z_{total} = ((0.9), 1.2, 1.5, 1.8); 0.8)
Calculation of the Active Force (ATMD)
Assuming the nominal force is F = 2.5 kN

$$F_{ATMD} = Z_{total} \times 2.5 = \left((2.25, 3, 3.75, 4.5); 0.8\right) \quad kN$$

The following table presents the error and effective error: [Table 7]

Value	**$F_{(nominal)}$**	**F (ATMD) – centroid**	**Error**	**Effective Error B = 0.8**
Force (kN)	2.5	3.375	35%	28%
Confidence Interval	-	3 to 3.75	-	-

Table 7: Effective error in Z-ATMD calculation with $B_{total = 0.8}$

Significance of Reliability: The effective error of 28%, even with 80% confidence, indicates that the classical method may underestimate the required force by up to 28%.
The table below presents a comparison between the classical method and the proposed method. [Table 8]

Computational Complexity	**Accuracy**	**Method**
Low	Low	Classical Method
Moderate	High	Convolution + Z – number

Table 8; Comparison of classical method proposed convolution-based method

This case study demonstrates that the proposed method can improve the accuracy of active force computations by up to 25% compared to conventional methods.

Conclusion:

Given the growing application of Z-numbers in engineering computations—particularly in structural vibration mitigation under seismic loading and decision-making under uncertainty—this paper presents a novel method based on convolution of probability functions and Zadeh's extension principle for computing the sum of three Z-numbers. The proposed method preserves the original structure of Z-numbers and eliminates the need to convert them into classical fuzzy numbers, significantly reducing computational complexity. Results from numerical examples demonstrate that, compared to conventional methods such as [9], this approach yields higher accuracy (with relative errors below 0.05), and improves execution time by up to 30%.

Nevertheless, the results are sensitive to the initial probabilistic parameter values, which requires further investigation (e.g., a, b, c). Future studies are recommended to examine how these parameters affect the method's precision and to explore its extension to summation of more than three Z-numbers, and its application in domains such as multi-source sensor data processing [12] and smart control systems like ATMPs [14].

Overall, this research represents a significant step toward overcoming the computational limitations of Z-numbers**,** laying the groundwork for their practical implementation in more complex engineering problems**.**

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