Semi Modified Alpha Power Weibull Distribution And Its Statistical Properties

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Abstract

A new probability distribution is developed in this study by adding an extra parameter to the existing Alpha power modified transformation technique. The proposed study employed Weibull distribution as a baseline to the new probability generator called Semi Modified Alpha Power Weibull Distribution (SMAPWD). Several important statistical properties were developed for the new distribution such as, quantile function, median, mode, order statistics, r^{th} moments and MGF. Maximum likelihood function estimation method was used to derive the estimates of the parameters. Two real data sets were applied to the proposed distribution and have a better fit as compare to the class of other distributions.

Keywords: Probability Distribution, Weibull Distribution, Alpha Power, Median, Statistical Properties

Introduction

Probability distribution is becoming a normal practice for researchers to improve and explore to new generation, while linking with modern technologies. Real life problems their analysis and complex data sets need the probability distributions accordingly from simplifying the classical distributions [1]. for achieving the purpose, we create new generators by adding some new parameters to the baseline distribution or merge the existing ones [2] and [3] Substituting a new parameter to the existing distribution. [4] proposed the T-X family of continuous distributions. [5] suggested beta generated distributions with beta as a parent distribution and cumulative distribution function (CDF) as a baseline of a continuous random variable. Later on, the beta transformation got replaced [6] with Kumaraswamy distribution. Univariate continuous distribution was constructed and reviewed generously by [7] for comparison purpose. A novel approach recently proposed by [8] called the alpha power transformation (APT) aiming the skewness into the baseline distribution by adding a new parameter in a continuous distribution. Described below:

$$F_{APT(x)} = \begin{cases} \dfrac{\alpha^{F(x)} - 1}{\alpha - 1} & if\, \alpha > 0, \alpha \neq 0 \\ \mathrm{F}(x) & if\, \alpha = 1 \end{cases} \qquad (1)$$

And the corresponding probability density function:

$$F_{APT(x)} = \begin{cases} \dfrac{\log \alpha}{\alpha - 1} \alpha^{\mathrm{F}(x)} f(x) & if\, \alpha > 0, \alpha \neq 0 \\ f(x) & if\, \alpha = 1 \end{cases} \qquad (2)$$

This generator provides two parameter alpha power distribution from single parameter exponential distribution.

The shape of pdf, survival function and hazard rate function were analyzed. [9] and [1] used the above generator to introduce a three-parameter alpha power distribution by restoring the two parameter Weibull distribution. The researchers have used this generator to create many distributions. Like: lindly distribution [10], inverse lindly distribution [11], generalized exponential distribution [12] etc.
Weibull distribution (WD) is the extended form of exponential distribution which is developed by Weibull [13]. the Weibull distribution is used in several areas including survival, reliability and engineering problems. The Hazard function of Weibull distribution is suitable for all type of real-life data depending upon the parameter values and its fluctuations. The Hazard function of Weibull distribution (WD) is not suitable for modeling human mortality and machine life because it does not convey the non-monotonic pattern of the model [14]. Several researchers developed many important variations of Weibull distributions to cover the deficiencies. Such as, Almalki and Yuan (2013) proposed new modified Weibull distribution. [15] introduced Gumbel-Weibull probability function. Exponentiated Weibull (EW) [16] , Additive Weibull (AW) [17] , Weibull Extension (WE) [18] , Generalized Modified Weibull (GMW) [19], Exponential Weibull (EW) [20] and beta Sarhan–Zain din MW distribution respectively. [21] and [22] suggested recent developments in the distribution theory to the researchers.

The pdf and CDF of Weibull distribution is as follows:

$$f(x)=\frac{\theta}{\lambda}\left(\frac{x}{\lambda}\right)^{\theta-1}e^{-\left(\frac{x}{\lambda}\right)^{\theta}},x\geq 0 \quad (3)$$

$$F(x)=1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}} \quad (4)$$

Proposed Distribution

The PDF and CDF of the new proposed distribution using Alpha Power Transformation Technique are given by

$$F(x)=\frac{F(x)(1-\alpha^{F(x)})}{(1-\alpha)} \qquad x>0,\alpha>0 \quad (5)$$

$$f(x)=\frac{f(x)[\alpha^{F(x)}\log(\alpha)F(x)-(1-\alpha^{F(x)})]}{(\alpha-1)} \qquad x>0,\alpha>0 \quad (6)$$

The F(x) and f(x) represents CDF and PDF of the baseline distribution.

Semi Modified Alpha Power Weibull (SMAPW) Distribution

The suggested method specified in equation (5) is applied to Weibull distribution and a new model known as Semi Modified Alpha Power Weibull (SMAPW) distribution is obtained. So, the pdf and CDF of Semi Modified Alpha Power Weibull (SMAPW) distribution is represented by

$$F(x)=\frac{\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)(1-\alpha^{\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)})}{(1-\alpha)} \quad (7)$$

$$f(x)=\frac{\frac{\theta}{\lambda}\left(\frac{x}{\lambda}\right)^{\theta-1}e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\left[\alpha^{\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)}\log(\alpha)\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)-(1-\alpha^{\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)})\right]}{(\alpha-1)} \quad (8)$$

The graph of PDF and CDF are mentioned below in Figure:1

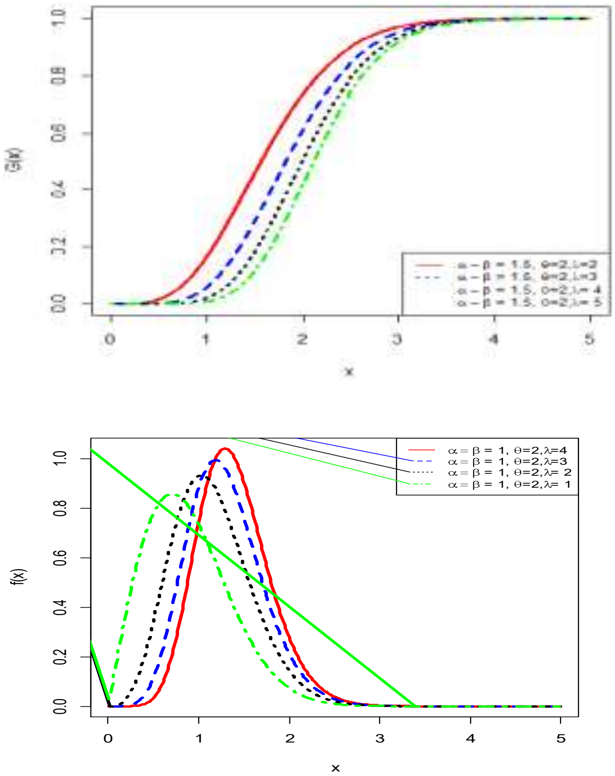

Figure 1: PDF and CDF of SMAPW Distribution

Survival and hazard rate function are as follows:

$$S(x) = 1 - F(x) = 1 - \frac{\left(1 - e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)\left(1 - \alpha^{\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)}\right)}{(1-\alpha)}$$

After simplification the survival function becomes:

$$S(x) = \frac{\alpha^{\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)}\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right) + e^{-\left(\frac{x}{\lambda}\right)^{\theta}} - \alpha}{1-\alpha} \tag{9}$$

$$H(x) = \frac{\text{pdf}}{\text{survival function}} = \frac{f(x)}{s(x)}$$

$$H(x) = \frac{\dfrac{\frac{\theta}{\lambda}\left(\frac{x}{\lambda}\right)^{\theta-1} e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\left[\alpha^{\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)}\log(\alpha)\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right) - \left(1-\alpha^{\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)}\right)\right]}{(\alpha-1)}}{\dfrac{\alpha^{\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)}\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right) + e^{-\left(\frac{x}{\lambda}\right)^{\theta}} - \alpha}{1-\alpha}}$$

The hazard function becomes:

$$H(x)=\frac{\frac{\theta}{\lambda}\left(\frac{x}{\lambda}\right)^{\theta-1}e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\left[\alpha^{\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)}\log(\alpha)\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)-\left(1-\alpha^{\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)}\right)\right]}{\alpha-\alpha^{\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)}\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}} \qquad (10)$$

The graph of SF and HF are specified below in Figure: 2

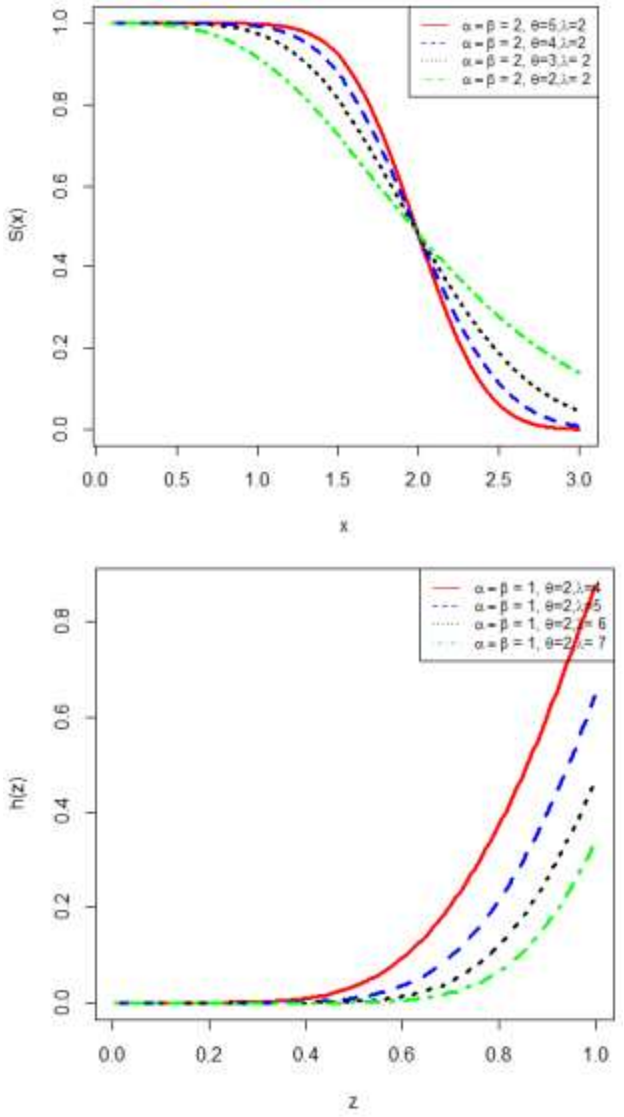

Figure:2 SF and HF of SMAPW Distribution

Quantile function

Quantile function is defined as an inverse of the distribution function.

$$F(X)=U$$
$$X=F^{-1}(U)$$

Where U follows the standard uniform distribution.

After simplification the quantile function is as follows:

$$X=e^{\left[\frac{1}{1-\theta}\left(log\left(log\left(\frac{log\ u(1-a)a}{log\ (a)}\right)+\theta log\ (\lambda)\right)-log(\theta)-\theta log\ (\lambda)\right)\right]} \qquad (11)$$

Median of the distribution is U=1/2 as follows:

$$Median = e^{\left[\frac{1}{1-\theta}\left(log\left(log\left(\frac{log\, a^{\frac{(1-\alpha)}{2}}}{log(\alpha)}\right)+\theta log\,(\lambda)\right)-log(\theta)-\theta log\,(\lambda)\right)\right]} \quad (12)$$

Mode

Mode of the distribution is derived as:

$$\frac{d}{dx}f(x)=0$$

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left[\frac{\frac{\theta}{\lambda}\left(\frac{x}{\lambda}\right)^{\theta-1}e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\left[\alpha^{\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)}log(\alpha)\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)-(1-\alpha^{\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)})\right]}{(\alpha-1)}\right]=0$$

After taking derivative mode of SMAPW transformation is:

$$\boldsymbol{Mode}=\frac{log(1-\alpha)\,log(\theta)}{\lambda} \qquad \alpha,\theta,\lambda>0 \qquad \textbf{(13)}$$

Order Statistics

Let $X_1,X_2,X_3, \ldots , X_n$ be the ordered random variables corresponding to a sample of size n. the the PDF of i[th] order statistics of MAPWD , is $f_{(i,n)}(X)$ given by the following expression

$$f_{(i,n)}(X)=\frac{n!}{(i-1)!\,(n-1)!}f(x)F(X)^{(i-1)}[1-F(X)]^{(n-i)}$$

By substituting eq (8) and eq (7) in eq i[th] order, we have

$$f_{(i,n)}(X)=\frac{n!}{(i-1)!\,(n-1)!}$$

$$\frac{\frac{\theta}{\lambda}\left(\frac{x}{\lambda}\right)^{\theta-1}e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\left[\alpha^{\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)}log(\alpha)\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)-\left(1-\alpha^{\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)}\right)\right]}{(\alpha-1)}\frac{\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)\left(1-\alpha^{\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)}\right)^{(i-1)}}{(1-\alpha)}$$

$$[1-\frac{\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)(1-\alpha^{\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)})}{(1-\alpha)}]^{(n-i)} \qquad \textbf{(14)}$$

Put $i=1$ in (14) to have expression of smallest order statistic.

$$f_{(1,n)}(X)=\frac{n!}{(n-1)!}\frac{\frac{\theta}{\lambda}\left(\frac{x}{\lambda}\right)^{\theta-1}e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\left[\alpha^{\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)}log(\alpha)\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)-\left(1-\alpha^{\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)}\right)\right]}{(\alpha-1)}\frac{\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)^{1}}{(1-\alpha)}\ [1-\frac{\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)(1-\alpha^{\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)})}{(1-\alpha)}]^{(n-1)}$$

Put $i = n$ in (14), we get largest order statistic as follows:

$$\boldsymbol{f}_{(\boldsymbol{n},\boldsymbol{n})}(\boldsymbol{X}) = \frac{n!}{(n-1)!\,(n-1)!}$$

$$\frac{\frac{\theta}{\lambda}\left(\frac{x}{\lambda}\right)^{\theta-1} e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\left[\alpha^{\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)} log(\alpha)\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)-\left(1-\alpha^{\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)}\right)\right]}{(\alpha-1)} \frac{\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)\left(1-\alpha^{\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)}\right)^{(\boldsymbol{n}-\boldsymbol{1})}}{(1-\alpha)}$$

Rth Raw Moment

$$\boldsymbol{\acute{\mu}_r = E(x)^r \int_0^{\infty} x^r f(x) dx}$$

$$\boldsymbol{\acute{\mu}_r = \frac{\theta^r}{(1-\alpha)}\left|\overline{\frac{r}{2}+1}\right.} \quad \textbf{(15)}$$

Which is the required rth raw moment.

Moment generating function

$$M_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(x) dx$$

$$M_x(t) = \int_0^{\infty} e^{tx}\left[\frac{\frac{\theta}{\lambda}\left(\frac{x}{\lambda}\right)^{\theta-1} e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\left[\alpha^{\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)} log(\alpha)\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)-(1-\alpha^{\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)})\right]}{(\alpha-1)}\right] dx$$

$$M_x(t) = \frac{\theta^r}{(1-\alpha)}\left|\overline{\frac{r}{2}+1}\right. \quad \textbf{(16)}$$

Is the required moment generating function.

Shannon and Entropy

$$\boldsymbol{S.E_x = -\int_0^{\infty} f(x) log f(x) dx}$$

The simplified form is as follows

$$\boldsymbol{S.E_X = -log[log(\alpha)\theta] + 1 - \alpha log(\alpha) + (\theta-1)log(\theta-1) - 1}$$

Simulations study

Simulation study has been performed for average MLEs, Mean Square Error (MSE) and bias. W= 100 samples of size n = 80, 120, and 160 were produced form EMAPP distribution. Random numbers were generated by the following expression

$$X = e^{\left[\frac{1}{1-\theta}\left(log\left(log\left(\frac{log\ u(1-a)a}{log\ (a)}\right)+\theta log\ (\lambda)\right)-log(\theta)-\theta log\ (\lambda)\right)\right]}$$

where U is uniform random numbers with parameter [0,1] Bias and MSE are calculated by

$$Bias = \frac{1}{W}\sum_{1=1}^{w}(\hat{b}_1 - b)$$

$$MSE = \frac{1}{W}\sum_{1=1}^{w}(\hat{b}_1 - b)^2$$

where b equals (α, θ, λ). Simulation results were obtained for various combinations of α, θ and λ. Table 1 displays the Average MSE and bias values. These estimations are consistent and close to genuine parameter values based on the sample size rises. Increasing sample size leads to lower MSEs and biases for all parameter combinations. The MLE technique accurately estimates SMAPW distribution characteristics.

Table:1 MSE and Bias

Parameters	N	MeanZ0	MeanZ1	MeanZ2	MSE0	MSE1	MSE2	BIAS0	BIAS1	BIAS2
A=2 B=3 C=4 w=100	80	1.9363	3.1465	5.9231	0.1911	6.4514	5.4267	-0.0457	-0.7362	1.4512
	120	1.9897	3.8130	4.8725	0.1304	4.2421	3.3476	-0.0302	-0.1898	0.6086
	160	1.9899	3.9623	4.5474	0.0459	2.9813	1.5937	-0.0215	-0.0587	0.3481

Applications

Two data sets have been analyzed to demonstrate the performance of the proposed SMAPW model. The first data set consists of cyst size in mm [23].

0.32	0.47	0.52	0.59	0.77	0.81	0.81	0.9	0.96	1.18
1.20	1.20	1.31	1.35	1.43	1.51	1.62	1.74	1.87	1.89
1.95	2.05	2.10	2.20	2.48	2.81	3.00	3.09	3.37	4.75

The second data set is related with the monthly actual taxes revenue in Egypt from January 2006 to November 2010.The data has been analyzed. The data values are as follows.

5.9	20.4	14.9	16.2	17.2	7.8	6.1	9.2	10.2	9.6	13.3	8.5	21.6	18.5	5.1
6.7	17	8.6	9.7	39.2	35.7	15.7	9.7	10	4.1	36	8.5	8	26.2	21.9
16.7	21.3	35.4	14.3	8.5	10.6	19.1	20.5	7.1	7.7	18.1	16.5	11.9	7	8.6
12.5	10.3	11.2	6.1	8.4	11	11.6	11.9	5.2	6.8	8.9	7.1	10.8		

The proposed distribution is compared with several other competitive models, like Weibull distribution, Modified Weibull distribution, exponentiated Kumaraswamy Inverse Weibull Distribution and Power Generalized Weibull Distribution with following pdf.

Weibull Distribution (WD)

$$f(x) = \frac{\theta}{\lambda}\left(\frac{x}{\lambda}\right)^{\theta-1} e^{-\left(\frac{x}{\lambda}\right)^{\theta}}, x \geq 0$$

Modified Weibull Distribution (MWD)

$$f(x) = (\alpha + \beta\gamma x^{\gamma-1})e^{-\alpha x-\beta x^{\gamma}} \quad : \quad X > 0$$

Exponentiated Kumaraswamy Inverse Weibull Distribution (EKIWD)

$$f(x) = \lambda\beta ab\theta(\lambda x)^{\beta-1} e^{-(\lambda x)^{\beta}} (1 - e^{-(\lambda x)^{\beta}})^{a-1} [1 - (1 - e^{-(\lambda x)^{\beta}})^{a}]^{b-1}$$

Power Generalized Weibull Distribution (PGWD)

$$f(x) = \frac{\alpha}{\beta\sigma^{\alpha}} x^{\alpha-1} \left[1 + \left(\frac{x}{\sigma}\right)\alpha\right]\frac{1}{\beta} - 1e^{1-[1+\left(\frac{x}{\sigma}\right)\alpha]\frac{1}{\beta}}$$

Using R software for Adequacy Model package, the goodness of fit test is used to evaluate the performance of SMAPW distribution and other Weibull distributions. Goodness of fit criteria include the result of Akaike's Information Criteria. There are several information criteria, including AIC, CAIC, BIC, and HQIC. Tables 2 and 3 show the Kulmogrov-Smirnov test (KS) results and p-values. A model is called good fit if it meets all of the criteria and has a higher p value.

Table: 2 Goodness of fit results data set 1

Distribution	MLE			AIC	CAIC	BIC	HQIC	P-value
SMAPWD	3.2398	-3.6863	4.3122	75.01439	75.93746	79.21798	76.35915	0.4809
WD	3.12158	2.7222		81.28669	81.73113	84.08908	82.1832	0.9987
EKIWD	2.7162	2.3043	3.1149	86.32938	88.82938	93.33536	88.57065	0.9911
PGWD	7.1234	3.7604	1.2351	82.32674	83.24981	86.53033	83.6715	0.9997
MWD	1.0541	-0.7323		84.20851	85.80851	89.8133	86.00153	1

TABLE:3 Goodness of fit results data set 2

Distribution	MLE			AIC	CAIC	BIC	HQIC	P-value
SMAPWD	12.6129	10.9867	1.9207	133.278	133.168	132.961	130.126	0.964
WD	0.5075	3.1962		216.439	216.529	216.539	518.217	1.367
EKIWD	16.8382	-0.492	13.1907	314.751	314.643	310.920	309.303	0.005
PGWD	27.5676	0.7695	1.0991	266.452	266.436	269.487	272.449	1.039e-08
MWD	8.9391	4.4233		294.981	294.972	492.891	487.935	3.3e-17

Tables 2 and 3 show that the SMAPW distribution has lower AIC, CAIC, BIC, HQIC, and log-likelihood values than alternative fitted distributions. Figures 3 and 4 shows best results for the proposed distribution also QQ-plot and PP-plot are included. Although certain QQ-plot values deviate from the proposed fitted distribution line, this is typical of heavy Right-tailed distributions [24].

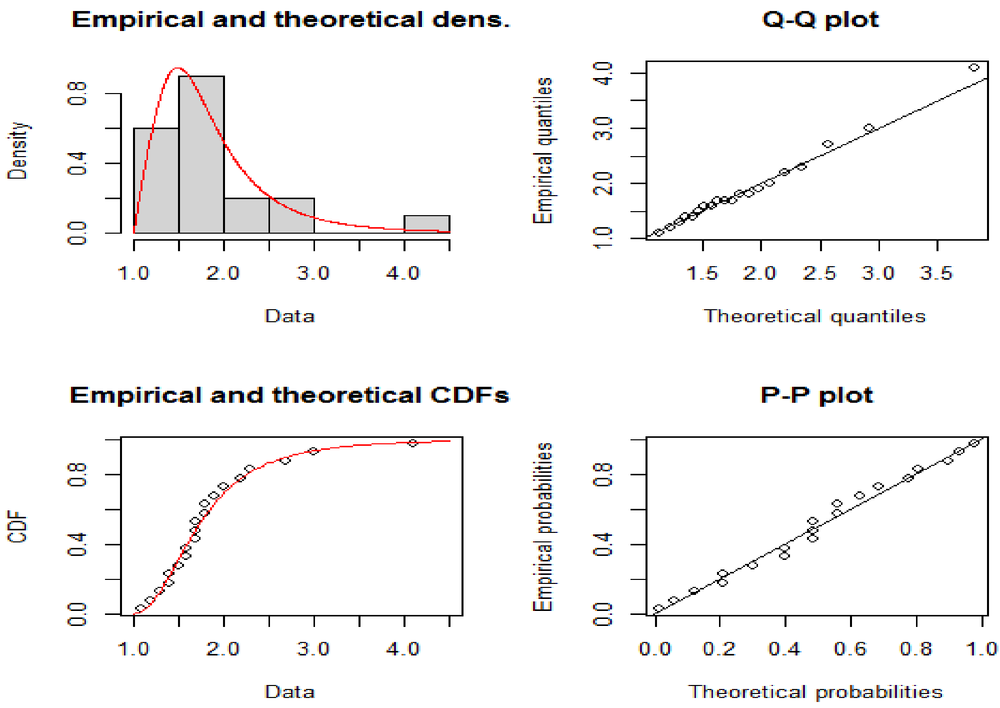

Figure 3 : Plots for data set 1

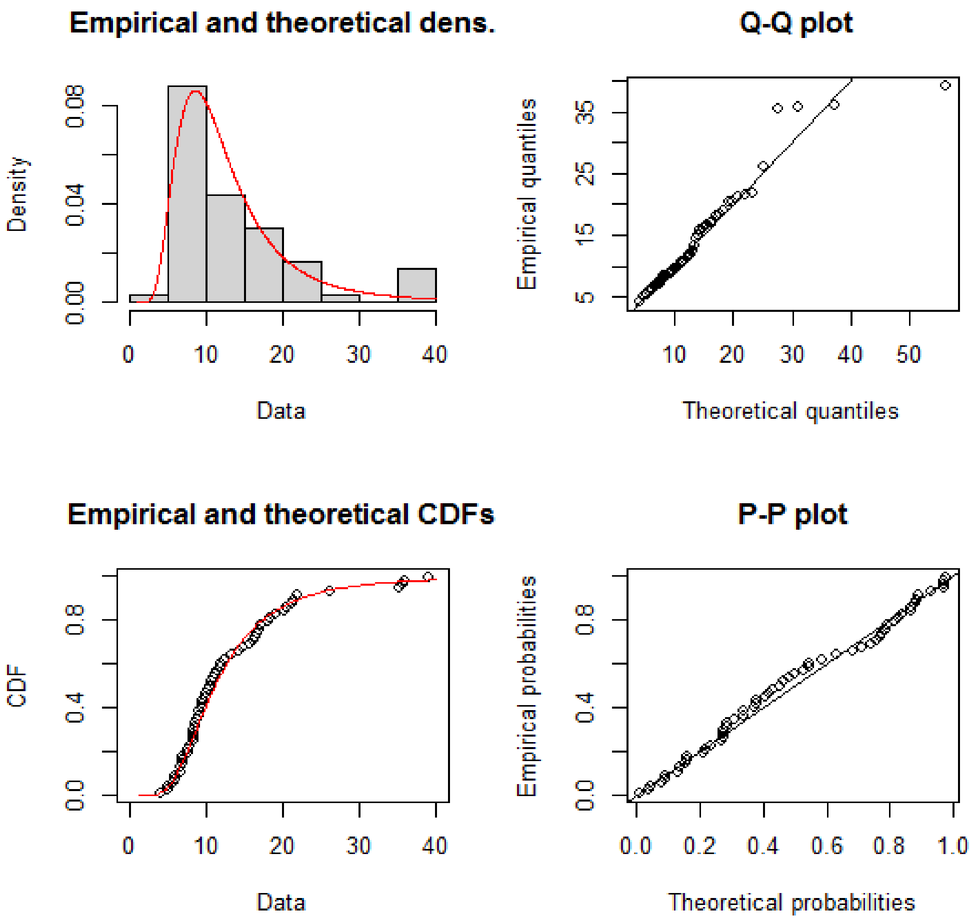

Figure 4: Plots for data set 2

Conclusion

The Proposed distribution, known as the SMAPW distribution, is introduced using alpha power Weibull distribution. The transformation adds skewness to a family of distribution functions. Various features of the distribution have been derived, including moment generation. Topics covered include probability function, order statistics, median, mode, MGF and Shannon entropy expressions. The maximum likelihood estimation approach was used to obtained the optimum unknown parameter estimates. The proposed distribution performed on other Weibull distributions on two real datasets and the results were satisfying.

References

1. Mudholkar G S, Srivastava DK. Exponentiated Weibull family for analyzing bathtub failure-rate data.IEEE transactions on reliability. 1993; 42(2):299–302.
2. Dey S, Sharma VK, Mesfioui M. A new extension of Weibull distribution with application to lifetime data.Annals of Data Science. 2017; 4(1):31–61.
3. Marshall AW, Olkin I. A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. Biometrika. 1997; 84(3):641–52.
4. Eugene N, Lee C, Famoye F. Beta-normal distribution and its applications. Communications in Statistics-Theory and methods. 2002; 31(4):497–512.
5. Jones M. C. Kumaraswamy's distribution: A beta-type distribution with some tractability advantages.Statistical Methodology. 2009; 6(1):70–81.
6. Alzaatreh A, Lee C, Famoye F. A new method for generating families of continuous distributions.Metron. 2013; 71(1):63–79.
7. Lee C, Famoye F, Alzaatreh AY. Methods for generating families of univariate continuous distributionsin the recent decades. Wiley Interdisciplinary Reviews: Computational Statistics. 2013; 5(3):219–38.
8. Mahdavi A, Kundu D. A new method for generating distributions with an application to exponential distribution.Communications in Statistics-Theory and Methods. 2017; 46(13):6543–57.
9. Nassar M, Alzaatreh A, Mead M, Abo-Kasem O. Alpha power Weibull distribution: Properties and applications.Communications in Statistics-Theory and Methods. 2017 18; 46(20):10236–52.
10. Dey S, Alzaatreh A, Zhang C, Kumar D. A new extension of generalized exponential distribution with application to Ozone data. Ozone: Science & Engineering. 2017; 39(4):273–85.
11. Dey S, Ghosh I, Kumar D. Alpha-Power Transformed Lindley Distribution: Properties and Associated Inference with Application to Earthquake Data. Annals of Data Science. 2018:1–28.
12. Hassan AS, Mohamd RE, Elgarhy M, Fayomi A. Alpha power transformed extended exponential distribution: properties and applications. Journal of Nonlinear Sciences and Applications. 2018; 12(4), 62–67.
13. Weibull, W. (1951). A statistical distribution function of wide applicability. Journal of applied mechanics.
14. Mudholkar, G. S., & Hutson, A. D. (1996). The exponentiated Weibull family: some properties and a flood data application. Communications in Statistics--Theory and Methods, 25(12), 3059-3083.
15. Al-Aqtash, R., Lee, C., & Famoye, F. (2014). Gumbel-weibull distribution: Properties and applications. Journal of Modern applied statistical methods, 13(2), 11.
16. Mudholkar, G. S., Srivastava, D. K., & Freimer, M. (1995). The exponentiated Weibull family: A reanalysis of the bus-motor-failure data. Technometrics, 37(4), 436-445.
17. Xie, M., & Lai, C. D. (1996). Reliability analysis using an additive Weibull model with bathtub-shaped failure rate function. Reliability Engineering & System Safety, 52(1), 87-93.
18. Carrasco, J. M., Ortega, E. M., & Cordeiro, G. M. (2008). A generalized modified Weibull distribution for lifetime modeling. Computational Statistics & Data Analysis, 53(2), 450-462.
19. Cordeiro, G. M., Hashimoto, E. M., & Ortega, E. M. (2014). The McDonald Weibull model. Statistics, 48(2), 256-278.
20. Saboor, A., Bakouch, H. S., & Khan, M. N. (2016). Beta sarhan–zaindin modified Weibull distribution. Applied Mathematical Modelling, 40(13-14), 6604-6621.
21. Lee, C., Famoye, F., & Alzaatreh, A. Y. (2013). Methods for generating families of univariate continuous distributions in the recent decades. Wiley Interdisciplinary Reviews: Computational Statistics, 5(3), 219-238.
22. Alzaatreh, A., Lee, C., & Famoye, F. (2014). T-normal family of distributions: a new approach to generalize the normal distribution. Journal of Statistical Distributions and Applications, 1(1), 1-18.
23. Hogg R. and Klugman S.A. Loss Distributions. New York: Wiley; 1984.
24. to Strength-Stress Models. American Journal of Mathematical and Management Sciences. 2018; 37 (1):80–92.