Performance Of Some New Quantile-Based Two Parameter Ridge Estimators For Linear Regression Model: Simulation And Application

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Abstract

In regression analysis, the efficiency of the ordinary least square (OLS) estimator decreases when the predictors become highly correlated leading to the problem of multicollinearity. In this study, new quantile- based two-parameter ridge (TPR) estimators are introduced to deal with the issue of multicollinearity in the linear regression model. The study presents a novel class of modified two-parameter ridge estimators developed using eigenvalues of the correlation matrix of predictors. The performance of the proposed TPR estimators is examined using extensive simulations using the mean squared error (MSE) criteria. The findings revealed that the TPR estimators have better performance than the one-parameter ridge estimators. In addition, the suggested estimator has superior performance than the OLS and is considered a one-parameter and two-parameter ridge estimator. Next, the application of the new TPR estimators is shown in the Economic Survey data. The findings indicate that the suggested NQW2 estimator outperformed all the competing estimators.

Keyword: Linear regression, OLS, multicollinearity MSE , ridge regression, TPR

1. Introduction

Considering of the linear regression (LR) model as follows:

$$y = X\beta + \varepsilon, \tag{1}$$

here y is represents the dependent variable vector, X is a predictors matrix, β is the vector of the unknown regression parameters, and ε denotes the error term vector. The ordinary least squares (OLS) technique minimizes the squared residuals to determine the estimated values of β. The OLS estimator and its covariance is defined as:

$$\hat{\beta} = (X'X)^{-1}X'y, \tag{2}$$

$$\text{Cov}(\hat{\beta}) = (X'X)^{-1}\ . \tag{3}$$

In linear regression, it is assumed that the predictors are uncorrelated with one another. But when this assumption is violated, the problem of multicollinearity exists within the X variables. To estimate the regression coefficient when multicollinearity is present, the efficiency of OLS estimators decreases leading to high variances which may develop invalid inferential statistics. To combat the multicollinearity issue, ridge regression was introduced by Hoerl and Kennard in 1970. They suggested adding a bias term (k) in the estimation of the regression coefficient to handle the problem of collinearity among X variables. Hoerl and Kennard (1970) defined the ridge regression (R) estimator as:

$$\hat{\beta}_R = (X'X + kI_p)^{-1} X'y, \qquad (2)$$

here $k > 0$ denotes the ridge parameter and I_P presents the identity matrix with order p. The p denotes the number of predictors in the model. To use ridge regression, it is necessary to obtain an optimal value of k.

Due to the significance of this issue, this study revisits the estimation issue of k. Many researchers have developed different approaches to estimating k (see Hockings et al. 1976, Kibria 2003; Liu 2003; Ali et al. 2021; Suhail et al. 2021; Ayinde et al. 2022; Shabbir et al. 2023; Wasim et al. 2023; Chand and Kibria, 2024; amongst others). In addition to the multicollinearity. Since ridge estimator's performance vary on basis of factors such as size of sample and the number of predictors. Determining an ideal value of k is complex and often involves both empirical and theoretical considerations. Due to the limitations, Lipovetsky and Conklin (2005) introduced a two-parameter ridge regression (TPR) approach. The TPR estimators are defined as (Lipovetsky and Conklin, 2005):

$$\hat{\beta}_{q,k} = q(X'X + kI)^{-1}X'y, \qquad (5)$$

where

$$q = \frac{(X'y)'(X'X + kI)^{-1}X'y}{(X'y)'(X'X + kI)^{-1}X'X(X'y)'(X'X + kI)^{-1}X'y}. \qquad (6)$$

The two-parameter ridge regression estimator improved the prediction accuracy of the regression coefficient when multicollinearity was present in the data. The value of q in Eq. (5) is determined by maximizing the coefficient of determination, while the k value minimizes the mean square error (MSE).

$$\hat{q}_{opt} = \frac{\sum_{i=1}^{p} \frac{\hat{\alpha}_i^2 \lambda_i}{\lambda_i + k}}{\sum_{i=1}^{p} \frac{\hat{\sigma}^2 \lambda_i + \hat{\alpha}_i^2 \lambda_i^2}{(\lambda_i + k)^2}}$$

The existing ridge regression estimators under specific conditions, do not uniformly perform well across all scenarios, especially in case of severe multicollinearity, high error variance, many predictors, and small sample sizes (Yasin et al., 2021). Many researchers have suggested methods of estimating the TPR estimators see for example, Tocker and Kaçıranlar (2013), Lukman et al. (2019), Şiray et al. (2021), Yasin et al. (2021). The room of estimating the optimal value of q and k parameter is still open. This serves as the basis for the development of new methods of estimating TPR estimator. This study also aims to develop new TPR estimators to deal with the issue of multicollinearity.

In this study, the quantile-based TPR estimator is developed to combat the issue of collinear predictors. Using different levels of quantile (low, moderate, and high), the estimator's performance is examined using intensive Monte Carlo simulations concerning minimum value of MSE. The simulation findings revealed the effectiveness of these estimators. It indicates that the new TPR estimator is efficient and robust in handling multicollinearity when error variance is high, the number of predictors is large, and a high to severe multicollinearity level exists in the data. The real-life data application on economic survey data in Pakistan is provided. The results showed that the new NQW2 estimator outperformed the OLS and considered a ridge estimator in the study.

The article proceeds with Section 2, which provides statistical methodology and a brief overview of some existing estimators. Section 3 provides information on the proposed estimators. Section 4 describes the simulation study conducted under various conditions. It also presents simulation results and discussion.

Section 5 displays the performance of the proposed estimators using real-world data. The study is concluded with final remarks in Section 7.

2. Methodology

In this section, the statistical methodology is provided with the discussion on the existing ridge estimators.

2.1. LR model in canonical form

The LR model in Eq. (1) can be expressed in canonical as follows:

$$y = Z\alpha + \varepsilon, \quad (7)$$

where $Z = XD$ $\alpha = D'\beta$ and $D'D = Ip$. D is an orthogonal matrix with eigen vectors that are $(X'X)$ matrix and Ip is an identity matrix. The term $\Lambda = D'X'XD$ such that $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_p)$. The Eq. (2) through Eq. (4) can be replicated in their canonical forms as follows:

$$\alpha_{OLS} = \Lambda Z'^{-1}y \quad (8)$$

$$\alpha_{(k)} = \left(\Lambda + kI_p\right)^{-1} Z'y \quad (9)$$

$$\alpha_{(q,k)} = q\left(\Lambda + kI_p\right)^{-1} Z'y \quad (10)$$

2.3. Some existing estimators

Hoerl and Kennard (1970a) suggested to estimate ridge estimator's k value as:

$$\hat{k}_{HK} = \frac{\hat{\sigma}^2}{\hat{\alpha}^2{}_i}, i = 1,2, \dots\dots, p \quad (11)$$

Hoerl and Kennard (1970b) suggested to estimate ridge estimator's k value as:

$$\hat{k}_{HK} = \frac{\hat{\sigma}^2}{\hat{\alpha}^2{}_{max}}, \quad (11)$$

where $\hat{\sigma}^2$ is the estimated error variance using OLS approach and $\hat{\alpha}_{max}$ is the largest of $\hat{\alpha}_1, \dots, \hat{\alpha}_p$.

Hoerl, Kennard, and Baldwin (1975) extended the idea of Hoerl and Kennard (1970b) and suggested k value to be estimated as:

$$\hat{k}_{HKB} = \frac{p\hat{\sigma}^2}{\sum_{i=1}^{p} \hat{\alpha}_i^2} \quad (12)$$

Kibria (2003) recommended to estimate ridge parameter k by taking the arithmetic mean, geometric mean and median of the ratio of $\hat{\sigma}^2$ and $\hat{\alpha}_i$ as follows:

$$\hat{k}_{KAM} = \frac{1}{P}\sum_{i=1}^{p} \frac{\hat{\sigma}^2}{\hat{\alpha}_i^2} \quad (13)$$

$$\hat{k}_{KGM} = \frac{\hat{\sigma}^2}{\left(\prod_{i=1}^{p} \hat{\alpha}^2\right)^{\frac{1}{p}}} \quad (14)$$

$$\hat{k}_{KMed} = Median\left(\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}\right) \quad (15)$$

Khalaf, Mansson, and Shukur (2013) developed the following estimator:

$$\hat{k}_{KMS} = \frac{\lambda_{max}}{\sum_{i=1}^{p}|\widehat{\alpha_i}|}\left\{\frac{\widehat{\sigma}^2}{\widehat{\alpha}_{max}^2}\right\}, \tag{16}$$

where λ_{max} is the maximum of $\lambda_1, \dots, \lambda_p$. $\frac{\lambda_{max}}{\sum_{i=1}^{p}|\widehat{\alpha}_i|}$

In order to enhance the one parameter ridge regression's fit quality, Lipovetsky and Conklin (2005) developed the two-parameter ridge estimator (LCTPR). They estimated the value of ridge parameter k and q using Eq. (11) and Eq. (5) respectively.

The two-parameter ridge estimator (TKTPR) was proposed by Toker and Kaciranlar (2013) developed two-parameter ridge (TPR) estimator using optimal values for q and k. The following formula will be used to get the ideal value of q:

$$\hat{q}_{opt} = \frac{\sum_{i=1}^{p}\frac{\widehat{\alpha}_i^2\lambda_i}{\lambda_i+k}}{\sum_{i=1}^{p}\frac{\widehat{\partial}^2\lambda_i+\widehat{\alpha}_i^2\lambda_i^2}{(\lambda_i+k)^2}} \tag{17}$$

In above equation (k) will be used from in Eq. (11). Then, after calculation of q_{opt} , k_{opt} will be calculated as below for TKTPR.

$$\hat{k}_{opt} = \frac{\hat{q}_{opt}\sum_{i=1}^{p}\widehat{\partial}^2\lambda_i+(\hat{q}_{opt}-1)\sum_{i=1}^{p}\widehat{\alpha}^2{}_i\lambda_i^2}{\sum_{i=1}^{p}\widehat{\alpha}^2{}_i\lambda_i^2} \tag{18}$$

The simulation study process and results are discussed in the next section.

3. New Proposed estimators

Building on the work of Liu (2003) Lipovetsky and Conklin (2005), we introduce six modified two-parameter Lipovetsky–Conklin ridge (MTPLCR) estimators to tackle the negative impact of multicollinearity in regression models effectively.

Khalaf (2013) proposed weight 'W' as:

$$W = \frac{\lambda_{max}}{\sum_{i=1}^{p}|\widehat{\alpha}_i|} \tag{19}$$

We multiplied equation (19) to equation (11) so we write:

$$NQW_i = W * HK = \frac{\lambda_{max}}{\sum_{i=1}^{p}|\widehat{\alpha}_i|} * \frac{\widehat{\sigma}^2}{\widehat{\alpha}^2{}_i}, i = 1,2,\dots\dots,p \tag{20}$$

Using the idea of Suhail et al. (2021), we will take the quantile of Eq. (20). Let $\{\hat{k}_{NQW_1}, \hat{k}_{NQW_2}, \hat{k}_{NQW_3} \dots\dots, \hat{k}_{NQW_p}\}$ be the realized values of $\hat{k}_{NQW_j}$ defined in Eq. (20). We write $\hat{k}_{NQW_1}, \hat{k}_{NQW_2}, \hat{k}_{NQW_3} \dots\dots, \hat{k}_{NQW_p}$ in ascending order of magnitude so that:

$$\hat{k}_{NQW_1} \leq \hat{k}_{NQW_2} \leq \hat{k}_{NQW_3} \leq \dots\dots\dots \leq \hat{k}_{NQW_p} \tag{21}$$

where $\hat{k}_{NQW_{(1)}} = \min(\hat{k}_{NQW_1}, \hat{k}_{NQW_2}, \hat{k}_{NQW_3} \dots\dots, \hat{k}_{NQW_p})$ and $\hat{k}_{NQW_{(p)}} = \max(\hat{k}_{NQW_1}, \hat{k}_{NQW_2}, \hat{k}_{NQW_3} \dots\dots, \hat{k}_{NQW_p})$ .then the set of $\{\hat{k}_{NQW_{(1)}}, \hat{k}_{NQW_{(2)}}, \hat{k}_{NQW_{(3)}} \dots\dots, \hat{k}_{NQW_{(p)}}\}$ is the order statistics for $(\hat{k}_{NQW_1}, \hat{k}_{NQW_2}, \hat{k}_{NQW_3} \dots\dots, \hat{k}_{NQW_p})$ and $\hat{k}_{NQW_{(j)}}, j = 1,2,3,\dots\dots, p$ is the j^{th} ordered observation. Now let NQW_γ where γ denotes the quantile level which ranges between 0 and 1 (i.e. $0 < \gamma < 1$). The $100\gamma^{th}$ quantile of $\{\hat{k}_{NQW_{(1)}}, \hat{k}_{NQW_{(2)}}, \hat{k}_{NQW_{(3)}} ...., \hat{k}_{NQW_{(p)}}\}$, then the new proposed quantile two parameter (NQW) estimators are given below:

$$NQW_{\gamma} = \{\hat{k}_{NQW_{(j)}}\}_{\gamma} = \{\hat{k}_{NQW_{(1)}}, \hat{k}_{NQW_{(2)}}, \hat{k}_{NQW_{(3)}}, \dots \dots \dots, \hat{k}_{NQW_{(p)}}\}_{\gamma} \quad (22)$$

Such that

$$p\left(\hat{k}_{NQW_{(j)}} < NQW_{\gamma}\right) = \gamma \quad (23)$$

There can be indefinite quantile points between 0 and 1. Following the idea of Suhail et al. (2021), we have considered different levels of γ i.e. low ($\gamma = 0.05, 0.25$), moderate ($\gamma = 0.50, 0.75$) and high ($\gamma = 0.95$ and 0.99). Therefore, the six new estimators will be denoted by NQW1, NQW2, NQW3, NQW4, NQW5 and NQW6. As per Dar and Chand (2024), we will consider the optimal choice of quantile probability according to data characteristics in our future study.

4. Simulation Study

4.1. Simulation technique

Following references Suhail et al. (2021) and Yasin et al. (2021), the predictors are generated as:

$$x_{ij} = (1-\rho^2)^{\frac{1}{2}} z_{ji} + \rho z_{ji} \qquad i = 1,2,, j = 1,2,., p$$

where ρ denotes the correlation level, n is size of sample, p is the number of predictors, and z_{ji} represents the random numbers generated normal distribution with zero mean and unit standard deviations. The dependent variable is generated using:

$$y = \beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p + \varepsilon_j; j = 1,2, \dots, n$$

In this study, the correlation level (ϕ), size of sample (n), the number of predictors (p), and error variance (σ^2) are varied in the simulation. The cases considered in the simulation study are given as follows:

Case 1: p=4, $n = 20, \sigma^2 = 0.1,1,5,10, \phi = 0.85,0.90,0.99,0.999$

Case 2: $p = 4, n = 60, \sigma^2 = 0.1,1,5,10, \phi = 0.85,0.90,0.99,0.999$

Case 3: $p = 4, n = 120, \sigma^2 = 0.1,1,5,10, \phi = 0.85,0.90,0.99,0.999$

Case 4: $p = 10, n = 20, \sigma^2 = 0.1,1,5,10, \phi = 0.85,0.90,0.99,0.999$

Case 5: $p = 10, n = 60, \sigma^2 = 0.1,1,5,10, \phi = 0.85,0.90,0.99,0.999$

Case 6: $p = 10, n = 120, \sigma^2 = 0.1,1,5,10, \phi = 0.85,0.90,0.99,0.999$

Following Kibria (2003), Suhail et al. (2021) and Dar et al. (2023) and the β represents the normalized eigen vector of the linked largest eigen value such that $\beta'\beta = 1$ and β_0 is assumed as zero. The mean square error (MSE) performance criteria examine estimator's performance. For any estimator ($\hat{\alpha}$) of α, the MSE is defined as (Chand and Kibria, 2024, Wasim et al. 2023 and Shabbir et al., 2023):

$$\text{MSE } (\hat{\alpha}) = \text{E}[(\alpha - \hat{\alpha})'(\alpha - \hat{\alpha})].$$

In this study, we have used 5000 replications, the estimated MSE (EMSE) is computed using:

$$MSE(\hat{\alpha}) = \frac{1}{5000}\sum_{i=1}^{5000}(\hat{\alpha}_{ij} - \alpha)'(\hat{\alpha}_{ij} - \alpha)$$

The simulation study is performed on R language (version 4.4.1). Tables 1–6 provide the results of simulations in terms of MSE each estimator under various scenarios. Table 7 provides a summary of the simulation study under different scenarios.

4.2. Simulation results and discussion

The simulation study results are provided in Tables 1-6. The following conclusions are based on the simulation outcomes:

i. In terms of minimal MSE for the majority of simulation scenarios, the suggested NQW6 estimator performs better for any fixed value of sample size, error variance, and number of predictor variables.
ii. Multicollinearity negatively impacts the efficiency of the OLS estimator. These findings are consistent with those reported in the literature; see, for example, Kibria (2003), Yasin et al. (2021), Wasim et al. (2023), and Chand and Kibria (2024).
iii. Generally, the OLS estimator and ridge estimator MSE values increases with the increase in the degree of collinearity while holding other factors constant. This is also reported in the study of Suhail et al. (2021) and Yasin et al. (2021).
iv. The MSE of the suggested estimators has an inverse relationship with the multicollinearity level. They demonstrate flexibility and robustness when multicollinearity increases from 0.85 to 0.999. However, the proposed estimator has a positive relationship between the estimated MSE and the error variance of an estimator.
v. For fixed values of n, Φ, and σ^2, the MSE of all the estimates increases with the rise in the p. These findings are in agreement with the results of Majid et al. (2022), Yasin et al. (2021), Ali et al. (2021), and Majid et al. (2022).
vi. For fixed values of n, p, and σ^2, the MSE values of all the estimators decrease with the increase in the size of the sample (n). These results are consistent with those published by Kibria (2003), Shabbir et al. (2023), and Dar et al. (2023). For all sample sizes, the ridge regression estimators perform well than the OLS estimator in the presence of multicollinearity performs far better than OLS estimators.

Table 1: Estimated MSE with different values of correlation level when $n = 20$ and $p = 4$

σ^2	**0.1**				**1**			
Φ	**0.85**	**0.90**	**0.99**	**0.999**	**0.85**	**0.90**	**0.99**	**0.999**
OLS	0.00782	0.02532	0.14440	1.54686	0.79942	2.57870	14.44415	156.8917
HK	0.00777	0.02476	0.12656	0.87781	0.52765	1.42882	7.77322	83.88854
HKB	0.00763	0.02333	0.10017	0.55043	0.36849	0.88270	4.20241	43.64493
KAM	0.19332	0.11441	0.06192	0.16176	0.29973	0.38603	0.85725	0.92011
KGM	0.00945	0.01198	0.04251	0.27330	0.22454	0.48250	1.63034	9.14228
KMED	0.04192	0.02041	0.03801	0.31351	0.26825	0.54786	1.39284	10.35134
KMS	0.00508	0.00791	0.00855	0.07604	0.17858	0.25567	3.91952	107.5088
LC	0.00777	0.02477	0.12659	0.87802	0.54288	1.45970	7.89659	85.10751
ST	953.486	244.717	3.80255	0.85398	3.79971	1.43919	1.28522	277.4191
NQW1	0.00014	0.00034	0.00508	0.05989	0.04897	0.14276	2.19331	75.34145

NQW2	0.00013	0.00013	0.00155	0.01760	0.02963	0.05971	1.02700	48.75538
NQW3	0.00012	0.00010	0.00013	0.00426	0.01627	0.02170	0.13286	11.34037
NQW4	0.00012	0.00010	0.00011	0.00056	0.01311	0.01266	0.01621	0.01379
NQW5	0.00012	0.00010	0.00011	0.00035	0.01294	0.01173	0.01204	0.01087
NQW6	**0.00012**	**0.00010**	**0.00011**	**0.00033**	**0.01293**	**0.01165**	**0.01183**	**0.01079**

σ^2	**5**				**10**			
Φ	**0.85**	**0.90**	**0.99**	**0.999**	**0.85**	**0.90**	**0.99**	**0.999**
OLS	19.8630	64.8760	353.712	3867.52	79.2583	258.8478	1450.8723	15908.4
HK	10.8962	35.0915	187.507	2059.41	43.0978	140.8831	782.5737	8670.2
HKB	6.1577	18.8227	97.5041	1062.68	23.7447	75.9467	410.3447	4559.8
KAM	1.33583	1.50389	1.15077	**0.48329**	**2.7029**	**2.8620**	**1.9432**	**1.1133**
KGM	2.49986	5.20045	15.1115	75.0002	7.1883	15.2111	44.5786	204.26
KMED	2.40551	5.40809	23.7600	254.818	8.1625	20.9047	102.1692	1032.5
KMS	6.53126	34.4997	279.320	3707.80	45.7410	196.0596	1329.0370	15694.8
LC	13.8863	43.7380	232.332	2532.94	61.3136	193.6436	1063.1648	11605.3
ST	3.46848	2.05921	259.988	401.598	6.7455	3.7178	14545.9730	132.072
NQW1	6.98558	31.2270	244.030	3410.85	50.1599	189.7788	1234.4680	15077.9
NQW2	4.32423	20.6727	185.076	3016.32	38.2553	150.7607	1071.6695	14188.2
NQW3	1.35827	5.44713	61.7397	1474.60	15.4041	58.6240	508.5570	8761.44
NQW4	0.54142	0.46169	0.50302	0.86337	4.4613	3.7035	2.5548	1.4654
NQW5	0.51219	0.43175	0.48131	0.85915	4.1648	3.4455	2.3621	1.4394
NQW6	**0.51014**	**0.43014**	**0.48053**	0.85903	4.1456	3.4325	2.3553	1.4387

Note: Bold values represent minimum MSE column-wise

Table 2: Estimated MSE with different values of correlation level when $n = 60$ and $p = 4$

σ^2	**0.1**				**1**			
Φ	**0.85**	**0.90**	**0.99**	**0.999**	**0.85**	**0.90**	**0.99**	**0.999**
OLS	0.003212	0.010068	0.055034	0.589657	0.316418	1.003689	5.467541	56.10742
HK	0.003204	0.009980	0.052391	0.400165	0.250933	0.613354	2.899327	29.34225
HKB	0.003179	0.009731	0.046515	0.272036	0.183505	0.404691	1.618974	14.91049
KAM	0.163517	0.097027	0.045045	0.076307	0.208392	0.206512	0.486849	0.896279
KGM	0.004006	0.005253	0.019908	0.131096	0.100760	0.212875	0.764631	3.944169
KMED	0.027655	0.012798	0.015309	0.139492	0.116098	0.257737	0.764807	3.625263
KMS1	0.001729	0.002185	0.001681	0.001925	0.198277	0.136448	0.227690	15.56658
LC	0.003204	0.009981	0.052394	0.400201	0.252871	0.618156	2.915352	29.47928
ST	11326.68	2777.632	69.13679	0.624112	13.90397	3.103166	1.085908	36.38591
NQW	0.000033	0.000032	0.000190	0.003382	0.004905	0.007912	0.100169	9.371182
NQW2	0.000033	0.000029	0.000047	0.001113	0.003885	0.004532	0.035457	4.469978

NQW3	0.000032	0.000029	0.000030	0.000121	0.003258	0.003260	0.005090	0.566330
NQW4	0.000040	0.000029	0.000030	0.000042	0.003187	0.002956	0.003204	0.003386
NQW5	0.000033	0.000029	0.000030	0.000036	0.003182	0.002937	0.003031	0.003110
NQW6	**0.000001**	**0.000029**	**0.000030**	**0.000036**	**0.003182**	**0.002936**	**0.003019**	**0.003101**

σ^2	**5**				**10**			
Φ	**0.85**	**0.90**	**0.99**	**0.999**	**0.85**	**0.90**	**0.99**	**0.999**
OLS	8.0019	25.2884	134.8083	1448.6529	31.3927	102.0309	535.9062	5744.2924
HK	4.2472	13.4083	70.7640	764.6428	16.4982	54.4939	282.1473	3021.4021
HKB	2.3782	7.0913	35.9878	387.1238	8.8365	28.6098	145.4466	1522.2135
KAM	0.7821	1.0657	0.8732	0.2724	1.3149	1.3903	0.9849	0.4270
KGM	1.1297	2.5219	7.5616	35.7600	3.0493	6.5996	20.6244	81.9932
KMED	1.0907	2.2571	9.1404	91.6434	3.0304	7.9669	38.7895	352.4658
KMS	0.6974	3.7108	57.0518	1182.2489	5.3876	37.4656	360.5455	5309.8299
LC	4.7484	14.8133	77.8839	843.2993	21.3071	68.9290	356.3632	3829.9336
ST	2.4389	1.6274	1.1912	39.9681	2.8327	1.8291	1.2172	18.7904
NQW1	0.4147	2.8603	42.9790	983.0377	6.9656	37.1377	324.4627	4778.5132
NQW2	0.2256	1.3456	24.3540	742.3264	4.0094	22.7286	237.4651	4085.4150
NQW3	0.0922	0.2490	4.4201	222.6900	1.0106	5.1577	72.8196	1767.0088
NQW4	0.0775	0.0750	0.0748	0.0772	0.4980	0.4525	0.3399	0.3661
NQW5	0.0766	0.0736	0.0739	0.0770	0.4928	0.4474	0.3372	0.3657
NQW6	**0.0765**	**0.0735**	**0.0739**	**0.0770**	**0.4924**	**0.4471**	**0.3371**	**0.3656**

Note: Bold values represent minimum MSE column-wise

Table 3: Estimated MSE with different values of correlation level when $n = 120$ and $p = 4$

σ^2	**0.1**				**1**			
Φ	**0.85**	**0.90**	**0.99**	**0.999**	**0.85**	**0.90**	**0.99**	**0.999**
OLS	0.001370	0.004289	0.024616	0.263952	0.136347	0.437960	2.380116	26.276547
HK	0.001368	0.004273	0.024065	0.209359	0.122946	0.323295	1.279402	13.766827
HKB	0.001364	0.004226	0.022634	0.152236	0.100212	0.227989	0.761791	6.992720
KAM	0.207198	0.122932	0.054386	0.041209	0.226911	0.166663	0.280519	0.861271
KGM	0.004490	0.003046	0.009733	0.065755	0.060328	0.116620	0.388232	2.242252
KMED	0.044036	0.015832	0.008102	0.061142	0.082715	0.129796	0.448661	1.817826
KMS	0.000973	0.001388	0.001232	0.000500	0.224454	0.181352	0.090883	3.014797
LC	0.001368	0.004273	0.024066	0.209370	0.123516	0.325023	1.284893	13.80869
ST	33432.41	12715.57	542.4546	0.789016	57.71392	9.452823	1.548154	1.044232
NQW1	0.000023	0.000020	0.000031	0.000620	0.002775	0.003200	0.009274	1.461321
NQW2	0.000022	0.000020	0.000020	0.000245	0.002451	0.002441	0.003941	0.572643

NQW3	0.000003	0.000020	0.000019	0.000026	0.002338	0.002007	0.002276	0.044344
NQW4	0.000022	0.000026	**0.000019**	0.000020	0.002327	0.001955	0.002005	0.002125
NQW5	0.000022	0.000020	**0.000019**	0.000020	0.002326	0.001952	0.001979	0.001989
NQW6	**0.000022**	0.000026	0.000019	**0.000020**	**0.002326**	**0.001951**	**0.001977**	**0.001983**

σ^2	**5**				**10**			
Φ	**0.85**	**0.90**	**0.99**	**0.999**	**0.85**	**0.90**	**0.99**	**0.999**
OLS	3.429570	10.79918	59.62768	639.31865	14.00412	43.86510	245.1011	2685.0705
HK	1.866440	5.636566	31.01300	329.82931	7.542811	23.24133	128.7528	1414.741
HKB	1.131173	3.033109	15.70035	164.18605	4.142835	12.24304	64.76089	708.1365
KAM	0.569524	0.813273	0.984602	0.367483	1.029115	1.228747	0.969294	0.379733
KGM	0.649538	1.330301	4.234720	20.511499	1.724918	3.659639	10.49246	50.92664
KMED	0.711521	1.212521	4.196595	39.362367	1.664502	3.586334	15.10803	160.0451
KMS	0.593242	0.639720	12.14114	400.64883	1.145750	7.371583	110.6865	2233.097
LC	2.049324	6.070936	33.15300	352.31976	9.431211	28.31074	156.0686	1702.072
ST	3.403793	2.011393	1.261902	1.458585	3.128647	1.971722	1.274934	1.097503
NQW1	0.092270	0.345216	8.217781	299.31793	1.274610	7.569382	92.11474	1901.472
NQW2	0.070086	0.163716	3.739307	188.80118	0.756486	3.905878	55.78551	1471.156
NQW3	0.059825	0.058321	0.589533	41.247121	0.353027	0.772876	10.44819	477.5583
NQW4	0.057561	0.050196	0.049095	0.049375	0.303199	0.260076	0.206327	0.200708
NQW5	0.057281	0.049737	0.048656	0.049281	0.301219	0.258144	0.205228	0.200519
NQW6	**0.057254**	**0.049702**	**0.048634**	**0.049277**	**0.301057**	**0.258022**	**0.205174**	**0.200511**

Note: Bold values represent minimum MSE column-wise

Table 4: Estimated MSE with different values of correlation level when $n = 20$ and $p = 10$

σ^2	**0.1**				**1**			
Φ	**0.85**	**0.90**	**0.99**	**0.999**	**0.85**	**0.90**	**0.99**	**0.999**
OLS	0.066090	0.225758	1.342127	15.008462	6.715339	23.08423	136.1847	1506.9065
HK	0.062464	0.185517	0.814766	8.391512	3.873387	13.16116	76.71978	842.9645
HKB	0.044747	0.097678	0.316636	2.641847	1.432063	4.420439	23.79825	253.1495
KAM	0.254207	0.141823	0.071495	0.107258	0.310837	0.321734	0.850685	3.551038
KGM	0.007443	0.013101	0.056663	0.469599	0.333977	0.866572	3.637340	29.85190
KMED	0.009004	0.012028	0.056594	0.567383	0.406117	1.104438	3.697273	34.30477
KMS	0.009874	0.012476	0.048919	2.980390	0.823610	5.972737	74.40309	1298.160
LC	0.062471	0.185538	0.814846	8.392067	3.911433	13.25305	77.17462	847.8308
ST	50.60472	2.037735	0.838757	17.40814	5.204748	4.164736	9244.205	19125.60
NQW1	0.001140	0.010009	0.064884	0.906399	0.450737	2.478172	33.44082	804.5871
NQW2	0.000065	0.000058	0.000501	0.050583	0.039118	0.149054	3.267768	168.3269

NQW3	0.000064	0.000050	0.000051	0.003474	0.008829	0.019072	0.282075	32.16717
NQW4	0.000064	0.000050	0.000045	0.000123	0.006727	0.006080	0.019369	2.562603
NQW5	0.000064	0.000050	0.000045	0.000045	0.006548	0.005196	0.004952	0.005686
NQW6	**0.000064**	**0.000050**	**0.000045**	**0.000045**	**0.006546**	**0.005190**	**0.004802**	**0.005114**

$\boldsymbol{\sigma^2}$	**5**				**10**			
$\boldsymbol{\Phi}$	**0.85**	**0.90**	**0.99**	**0.999**	**0.85**	**0.90**	**0.99**	**0.999**
OLS	170.1519	587.1070	3339.836	37430.62	665.9246	2352.601	13265.19	147649.4
HK	97.45487	336.5571	1883.783	20919.08	380.5246	1343.612	7365.147	82034.09
HKB	32.83343	109.3022	595.4714	6295.077	129.8473	434.9593	2275.833	24336.88
KAM	1.461837	2.497128	4.260751	2.302943	3.360224	4.646884	5.293950	2.131803
KGM	5.849687	15.55151	64.66582	516.9890	20.42503	52.90913	223.7390	1648.686
KMED	6.403397	17.64854	83.79544	862.0775	25.35880	72.19713	351.4275	3336.929
KMS	102.9152	453.4375	3063.270	36953.25	533.8132	2120.153	12873.59	147066.5
LC	107.4082	365.2359	2032.029	22523.09	438.6039	1522.941	8302.924	92000.11
ST	3.980018	13.78397	14406.43	3377.835	7.473416	13.60770	11987.10	150368.6
NQW1	70.12354	308.7818	2384.467	34632.49	431.5163	1713.390	11664.81	143942.8
NQW2	13.42105	73.38040	797.9542	20921.87	141.3010	633.7510	5899.512	115944.4
NQW3	2.144055	13.67132	205.5586	7282.241	44.11745	206.2995	2101.319	55524.16
NQW4	0.337467	1.214184	24.92807	1501.194	9.057734	38.68742	481.8980	14858.13
NQW5	0.194355	0.158255	0.116592	0.113027	3.261036	2.927190	1.757013	0.651590
NQW6	**0.193322**	**0.156558**	**0.114256**	**0.112334**	**3.249233**	**2.916585**	**1.746599**	**0.649244**

Note: Bold values represent minimum MSE column-wise

Table 5: Estimated MSE with different values of correlation level when $n = 60$ and $p = 10$

$\boldsymbol{\sigma^2}$	**0.1**				**1**			
$\boldsymbol{\Phi}$	**0.85**	**0.90**	**0.99**	**0.999**	**0.85**	**0.90**	**0.99**	**0.999**
OLS	0.0095592	0.0305241	0.1668309	1.8489797	0.9484118	3.0789936	16.9022915	189.549372
HK	0.0095193	0.0300297	0.1508665	1.1924296	0.7165867	1.9433023	10.0133193	111.946457
HKB	0.0091834	0.0265138	0.0972062	0.5348274	0.3701181	0.8742598	3.8768423	40.0730969
KAM	0.2848776	0.1650404	0.0747257	0.0508436	0.3130843	0.2253224	0.3158612	1.4906090
KGM	0.0059769	0.0046287	0.0174880	0.1407816	0.1079943	0.2640046	1.0679573	7.9434923
KMED	0.0101488	0.0046318	0.0174805	0.1770670	0.1315527	0.3557414	1.5043966	10.0524035
KMS	0.0039270	0.0045775	0.0034086	0.0219061	0.1324519	0.0985740	1.5281234	84.1798622
LC	0.0095197	0.0300310	0.1508737	1.1924974	0.7220241	1.9557208	10.057830	112.373490
ST	4041.7982	415.72247	3.3534111	0.8862208	17.335189	9.4086154	3.1289785	1989.15079
NQW1	0.0000321	0.0000517	0.0015650	0.0253726	0.0170259	0.0407013	0.9009181	54.2761750
NQW2	0.0000301	0.0000222	0.0000216	0.0005669	0.0035924	0.0053660	0.0601530	10.8327773

NQW3	0.0000301	0.0000222	0.0000205	0.0000322	0.0031586	0.0024077	0.0066370	0.9954213
NQW4	0.0000301	0.0000221	**0.0000204**	0.0000208	0.0031105	0.0021904	0.0022375	0.0332074
NQW5	**0.0000301**	**0.0000221**	0.0000205	**0.0000202**	0.0031040	0.0021685	0.0019344	0.0021382
NQW6	**0.0000301**	**0.0000221**	0.0000205	**0.0000202**	**0.0031039**	**0.0021683**	**0.0019324**	**0.0020952**

σ^2	**5**				**10**			
Φ	**0.85**	**0.90**	**0.99**	**0.999**	**0.85**	**0.90**	**0.99**	**0.999**
OLS	23.67065	76.97940	422.6012	4605.289	96.98399	306.3641	1699.230	18791.83
HK	14.38918	46.34017	250.3295	2689.630	59.31776	183.6340	1005.314	11032.53
HKB	5.895708	18.09227	93.57325	974.1668	23.76507	70.97095	370.6543	3957.637
KAM	0.720123	1.153754	2.339344	1.866205	1.522672	2.324677	3.041170	1.443465
KGM	1.803190	4.479258	18.02363	136.7213	5.899269	14.92281	59.49627	445.3506
KMED	2.389279	5.580745	26.14115	260.6541	7.950690	21.57058	101.7194	1040.774
KMS	2.869425	21.17709	248.8581	4141.107	32.18638	163.1445	1369.539	18178.23
LC	16.04202	50.46191	270.0667	2888.952	72.92014	219.1232	1177.856	12771.13
ST	6.275401	4.052019	1.879493	177.7177	8.379641	4.838079	2.077150	1220.934
NQW1	3.134676	20.35058	216.3308	3846.581	42.39665	175.8468	1301.370	17800.16
NQW2	0.442611	3.491780	72.51704	2190.082	13.56149	70.95951	700.0765	13620.15
NQW3	0.109521	0.312159	11.59246	781.7943	2.635721	14.09922	229.4251	7090.097
NQW4	0.078741	0.068174	0.691801	97.02846	0.815526	1.696566	30.16853	1656.846
NQW5	0.074716	0.054019	0.050026	0.050334	0.700834	0.433686	0.400038	0.210911
NQW6	**0.074662**	**0.053911**	**0.049843**	**0.050248**	**0.700187**	**0.432906**	**0.399318**	**0.210702**

Note: Bold values represent minimum MSE column-wise

Table 6: Estimated MSE with different values of correlation level when $n = 120$ and $p = 10$

σ^2	**0.1**				**1**			
Φ	**0.85**	**0.90**	**0.99**	**0.999**	**0.85**	**0.90**	**0.99**	**0.999**
OLS	0.004239	0.013789	0.075971	0.837699	0.423936	1.395531	7.559456	82.58086
HK	0.004232	0.013694	0.072595	0.598039	0.364634	0.971558	4.473914	48.19560
HKB	0.004169	0.012926	0.054966	0.290277	0.213478	0.468028	1.819084	17.54303
KAM	0.281794	0.162605	0.073298	0.033583	0.295865	0.193498	0.185498	0.944462
KGM	0.004110	0.002447	0.008258	0.065482	0.055900	0.126782	0.523214	4.089420
KMED	0.008130	0.002634	0.007652	0.075778	0.063523	0.163299	0.774700	4.827877
KMS	0.001831	0.002008	0.001249	0.001423	0.176580	0.102224	0.124795	13.34055
LC	0.004232	0.013695	0.072596	0.598055	0.365724	0.974492	4.484084	48.28154
ST	22700.03	2773.995	39.21009	0.687608	38.10752	19.41720	5.415624	82.63389
NQW1	0.000014	0.000010	0.000087	0.004183	0.002881	0.006114	0.058327	8.184859
NQW2	0.000014	0.000010	0.000009	0.007655	0.001398	0.001154	0.003312	0.717728
NQW3	0.000014	0.000010	0.000009	**0.000006**	0.001365	0.000988	0.001048	0.044001

NQW4	0.000014	0.000010	0.000009	0.000089	0.001361	0.000975	0.000859	0.001817
NQW5	0.000014	0.000009	0.000009	0.000009	0.001361	0.000973	0.000844	0.000878
NQW6	**0.000013**	**0.000000**	**0.000001**	0.000009	**0.001360**	**0.000973**	**0.000844**	**0.000873**

σ^2	**5**				**10**			
Φ	**0.85**	**0.90**	**0.99**	**0.999**	**0.85**	**0.90**	**0.99**	**0.999**
OLS	10.7101	34.4244	189.890	2085.36	42.8176	138.169	777.479	8406.73
HK	6.54438	20.6987	112.103	1228.29	26.1733	83.0598	463.798	4952.96
HKB	2.75809	8.08692	41.9070	443.381	10.5665	31.9333	171.635	1784.41
KAM	0.49569	0.67252	1.66333	2.23450	0.96115	1.51725	2.68857	1.42660
KGM	0.86078	2.15872	8.98243	67.1425	2.92058	7.24982	30.1389	220.184
KMED	1.28582	2.74348	11.5766	114.769	3.73787	9.68409	45.9695	455.725
KMS	0.43567	2.59311	53.2145	1499.25	4.08696	31.6896	427.939	7423.70
LC	6.97385	21.6987	116.810	1275.34	30.4619	94.2694	517.855	5496.14
ST	7.75724	4.98917	2.06304	13893.1	6.55099	3.97818	1.80773	2015.87
NQW1	0.24279	2.29738	44.6897	1259.03	5.90001	36.8436	397.978	6943.18
NQW2	0.05014	0.17547	6.83579	468.232	0.77057	5.94033	125.024	3896.42
NQW3	0.03593	0.03399	0.50574	89.0299	0.21225	0.56193	17.3893	1329.39
NQW4	0.03362	0.02565	0.03869	6.22409	0.16801	0.11577	0.94398	168.454
NQW5	0.03324	0.02489	0.02250	0.02212	0.16320	0.10150	0.08688	0.08450
NQW6	**0.03323**	**0.02488**	**0.02247**	**0.02210**	**0.16314**	**0.10142**	**0.08677**	**0.08447**

Note: Bold values represent minimum MSE column-wise

Table 7 shows the summary of the recommended estimators based on different cases of the simulation study about the minimum value of MSE. It shows that the suggested NQW6 has the best performance in most situations. However, at a small sample (n =20), high error variance (σ^2 =10), and low dimensions (p=4), the KAM estimator has the best performance for all levels of collinearity.

Table 7: Summary of the simulation study in terms of minimum MSE values.

Φ		**0.85**	**0.95**	**0.99**	**0.999**	**0.85**	**0.95**	**0.99**	**0.999**
n	σ^2	$p = 4$				$p = 10$			
20	0.1	NQW6	NQW6	NQW6	NQW6	NQW6	NQW6	NQW6	NQW6
	1	NQW6	NQW6	NQW6	NQW6	NQW6	NQW6	NQW6	NQW6
	5	NQW6	NQW6	NQW6	KAM	NQW6	NQW6	NQW6	NQW6
	10	KAM	KAM	KAM	KAM	NQW6	NQW6	NQW6	NQW6
60	0.1	NQW6	NQW6	NQW6	NQW6	NQW4	NQW4	NQW4	NQW5
	1	NQW6	NQW6	NQW6	NQW6	NQW6	NQW6	NQW6	NQW6
	5	NQW6	NQW6	NQW6	NQW6	NQW6	NQW6	NQW6	NQW6
	10	NQW6	NQW6	NQW6	NQW6	NQW6	NQW6	NQW6	NQW6
120	0.1	NQW5	NQW5	NQW5	NQW6	NQW6	NQW6	NQW6	NQW3

1	NQW6	NQW6	NQW6	NQW6	NQW6	NQW6	NQW6	NQW6
5	NQW6	NQW6	NQW6	NQW6	NQW6	NQW6	NQW6	NQW6
10	NQW6	NQW6	NQW6	NQW6	NQW6	NQW6	NQW6	NQW6

5. Applications

To illustrate the performance of the new two-parameter ridge estimator, the application of the Pakistan economic survey 2017-2018 is provided using Livestock population and production data (Yasin et al., 2021). The data contains eighteen observations with five predictors. The MLR model is expressed as:

$$y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_4 + \beta_5 X_5 + \varepsilon_j \quad ; j = 1, \dots, n$$

where y denotes the livestock hairs production (tones), X_1, X_2, X_3, X_4 and X_5 denotes the number of buffaloes, cattle, goats, sheep's and poultry measured in millions.

Table 8 shows the correlations between dependent variable and predictors. It shows a positive and strong correlation between all the variables. It indicates the presence of multicollinearity in the livestock production data. The eigen values $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and λ_5 are found to be 4.8869,0.0777,0.0222, 0.0131 and 0.00003 respectively. The condition number of livestock data is 143734.3, which confirms the existence of multicollinearity in the data.

Table 8 This is a correlation matrix of Livestock production data.

	X_1	X_2	X_3	X_4	X_5	y
X_1	1	0.9931	0.9865	0.9866	0.9872	0.9897
X_2	0.9931	1	0.9611	0.9975	0.9760	0.9686
X_3	0.9865	0.9611	1	0.9469	0.9860	0.9982
X_4	0.9866	0.9975	0.9469	1	0.9589	0.9542
X_5	0.9872	0.9760	0.9860	0.9589	1	0.9909
y	0.9897	0.9686	0.9982	0.9542	0.9909	1

Table 9 shows the estimated MSE value and regression coefficient on the Livestock data. The KAM and KGM estimators are close competitors of the suggested NQW1, NQW2, and NQW3 estimators. The findings show that the suggested NQW2 estimator outperformed all the competing estimators with the minimum MSE (see Figure 1).

Table 9: Estimated MSE and regression coefficients of livestock production (2017-2018)

Estimators	MSE	$\hat{k}$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$	$\hat{\beta}_5$
OLS	78.5478	-	-0.4463	0.5096	-0.251	-0.8866	-0.5625
HK	1.2126	0.002801	-0.4463	0.5085	-0.2481	-0.3469	-0.0992
HKB	1.1858	0.006778	-0.4462	0.5069	-0.2442	-0.1861	-0.0457
KAM	1.1741	0.012851	-0.4462	0.5044	-0.2384	-0.1089	-0.0251
KGM	1.1794	0.009142	-0.4462	0.5059	-0.2419	-0.1458	-0.0346
KMED	1.1809	0.008478	-0.4462	0.5062	-0.2426	-0.1553	-0.0372
KMS	1.1809	0.00847	-0.4462	0.5062	-0.2426	-0.1554	-0.0372
LC	1.2126	0.002801	-0.4463	0.5085	-0.2481	-0.3469	-0.0992
ST	1.1894	0.002801	-0.3664	1.4635	0.0767	0.0015	0.0003
NQW1	1.1843	0.007249	-0.4463	0.5068	-0.2438	-0.1764	-0.0430
NQW2	**1.1733**	**0.021993**	**-0.4464**	**0.5011**	**-0.2304**	**-0.0671**	**-0.0149**
NQW3	1.1809	0.032643	-0.4465	0.4971	-0.2216	-0.0464	-0.0102
NQW4	1.2035	0.055514	-0.4467	0.4887	-0.2049	-0.0279	-0.006
NQW5	1.3555	0.455221	-0.448	0.3769	-0.0886	-0.0035	-0.0007
NQW6	1.3646	0.535163	-0.4481	0.3605	-0.0796	-0.003	-0.0006

Note: Bold represent estimator with minimum value of MSE

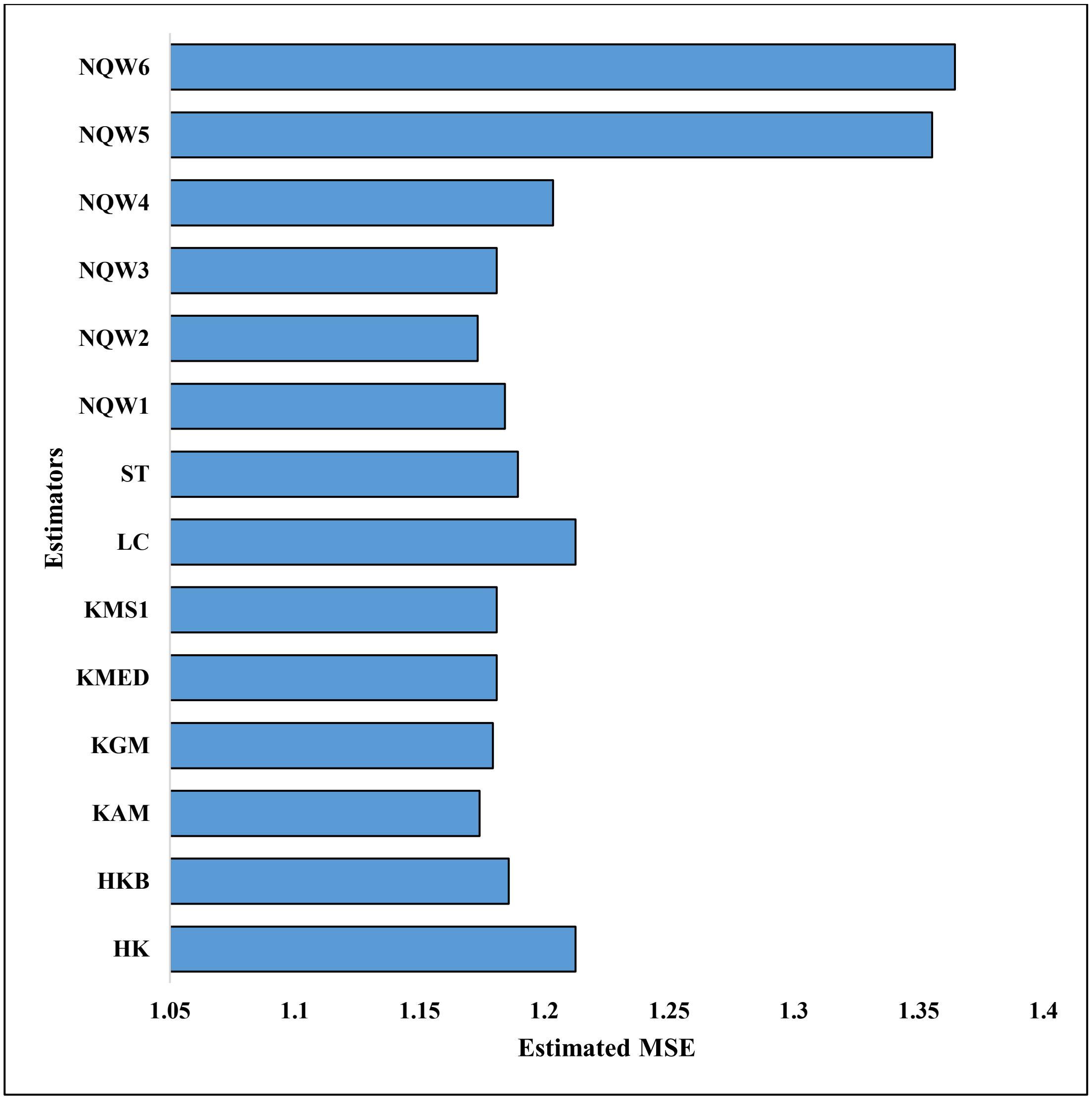

Figure 1: Estimated MSE of the ridge estimators

6. Some concluding remarks

This paper suggests quantile-based versions of two-parameter ridge (TPR) estimators that can be employed to deal with the issue of collinearity in linear regression model. The new TPR estimator's performance is examined using simulation study and real data analysis in terms of mean squared error (MSE). The suggested estimators are compared to the OLS, one-parameter and TPR estimators. Based on minimal MSE, the simulation results indicate that the new NQW6 estimator is outperformed other competing estimators in the study. In addition to NQW5, the NQW4, NQW3, and NQW2 estimators all performed much better in terms of minimal MSE and were in close competition with NQW6. Finally, the application on real data showed that the suggested estimators have superior performance than OLS and existing one-parameter and two-parameter ridge estimators.

References

1. Ali, S., Khan, H., Shah, I., Butt, M. M., & Suhail, M. (2021). A comparative analysis of modern and traditional robust ridge regression estimators. Communications in Statistics - Simulation and Computation, 50(8), 2213–2231. https://doi.org/10.1080/03610918.2019.1597119

2. Ayinde, K., Adewuyi, E., & Folaranmi, L. A. (2022). Exploring alternative ridge parameters in linear models. Nicel Bilimler Dergisi, 4(1), 22–46. https://doi.org/10.51541/nicel.1042316
3. Chand, S., & Kibria, B. M. G. (2024). A new ridge type estimator and its performance for the linear regression model: Simulation and application. Hacettepe Journal of Mathematics and Statistics, 1-14.
4. Dar, I. S., Chand, S., Shabbir, M., & Kibria, B. M. G. (2023). Condition-index based new ridge regression estimator for linear regression model with multicollinearity. Kuwait Journal of Science, 50(2), 91-96. https://doi.org/10.1016/j.kjs.2023.02.013
5. Hoerl, A. E., & Kennard, R. W. (1970). Ridge regression: Biased estimation for nonorthogonal problems. Technometrics, 12(1), 55-67.
6. Hocking, R. R., Speed, F. M., & Lynn, M. J. (1976). Introducing a class of biased estimators in linear regression. Technometrics, 18(4), 425-437. https://doi.org/10.1080/00401706.1976.10489466
7. Khalaf, G., Månsson, K., & Shukur, G. (2013). Proposing modified ridge regression estimators. Communications in Statistics - Theory and Methods, 42(8), 1476-1487. https://doi.org/10.1080/03610926.2011.649739
8. Kibria, B. M. G. (2003). Assessing the performance of new ridge regression estimators. Communications in Statistics - Simulation and Computation, 32(2), 419-435. https://doi.org/10.1081/SAC-120017499
9. Liu, K. (2003). Liu-type estimator as a solution to collinearity in regression models. Communications in Statistics - Theory and Methods, 32(5), 1009-1020. https://doi.org/10.1081/STA-120020406
10. Lukman, A. F., Ayinde, K., Siok Kun, S., & Adewuyi, E. T. (2019). A modified new two-parameter estimator in a linear regression model. Modelling and Simulation in Engineering, 2019(1), Article 6342702. https://doi.org/10.1155/2019/6342702
11. Shabbir, M., Chand, S., & Iqbal, F. (2023). A new ridge estimator for linear regression model with some challenging behavior of error term. Communications in Statistics - Simulation and Computation, 1-11.
12. Suhail, M., Chand, S., & Kibria, B. M. G. (2021). Quantile-based robust ridge m-estimator for linear regression model in presence of multicollinearity and outliers. Communications in Statistics: Simulation and Computation, 50(11), 3194–3206. https://doi.org/10.1080/03610918.2019.1621339
13. Toker, S., & Kaçıranlar, S. (2013). Evaluating the performance of a two-parameter ridge estimator under the mean square error criterion. Applied Mathematics and Computation, 219(9), 4718-4728. https://doi.org/10.1016/j.amc.2012.11.017
14. Üstündağ Şiray, G., Toker, S., & Özbay, N. (2021). Mathematical programming evidence for defining a two-parameter estimator. Journal of Statistical Computation and Simulation, 91(11), 2133–2152. https://doi.org/10.1080/00949655.2020.1834808
15. Wasim, D., Khan, S. A., & Suhail, M. (2023). Developing modified robust ridge m-estimators for linear regression models: Application to tobacco data. Journal of Statistical Computation and Simulation, 93(15), 2703-2724. https://doi.org/10.1080/00949655.2023.2207915
16. Yasin, S., Kamal, S., & Suhail, M. (2021). Performance of some new ridge parameters in two-parameter ridge regression model. Iranian Journal of Science and Technology, Transactions A: Science, 45(1), 327-341.