Identifying the Frauds in Digitally Consummated Transactions by Developing the frame work of Intuitionistic Fuzzy Graphs

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Abstract: In many branches of computer science and computational intelligence, a graph structure provides a helpful framing network for resolving combinatorial issues. Several concepts of intuitionistic fuzzy graph (IFG) structure are defined in this research piece by applying the idea of intuitionistic fuzzy sets to the graph structure. The study of different structures, such as IFGs, signed graphs, and graphs with labelled edges, can greatly benefit from the use of IFG structure. The degree and total degree of a vertex in the K-Lexicographic max product (K-LMP), as well as the IFG structure, are introduced, along with their respective notations. Additionally, numerical examples clarify the suggested concept. In instance, decision-making processes including the use of IFGs for the identification of fraudulent activity in online transactions are demonstrated. In conclusion, an algorithm explains the overall process of these applications.

Keywords: Intuitionistic fuzzy graph, K-Lexicographic max product, Degree, Total degree of a vertex, Decision making problems.

1. Introduction

1.1. Fuzzy Graphs

Since Zadeh's work [1] in 1965 developed the idea of partial truth between absolute true and absolute false, a fuzzy set may be thought of as a superset of a crisp set. The chemical industry, telecoms, decision-making, networking, computer science, and discrete mathematics are just a few of the areas that have found use for Zadeh's great idea. A fuzzy subgroup of a group is how Rosenfeld [2] first presented the idea of a fuzzy subset of a set. The development of fuzzy abstract algebra was made possible by Rosenfeld's article.

A graph depicts the interaction between vertices and edges and is a mathematical description of a network. Graph theory is used to describe real-world phenomena, but occasionally natural ambiguity about certain system properties prevents graphs from accurately representing a wide range of occurrences. The fuzzy graphs were defined with inspiration from a variety of real-world circumstances in [3]. Kauffman [4] used Zadeh's fuzzy relation [5] to introduce fuzzy graphs. Planning, scheduling, networking, data mining, clustering, picture capture, image segmentation, and many more fields are among the many fields in which fuzzy-graph theory is finding wide-spread applications. Lexicalographic products, or lexicographic min- and max-products of two fuzzy graphs, are described by Radha and Arumugam [6]. These concepts are comparable to the notion of lexicographic product in crisp graph theory. The maximum and semistrong products of fuzzy network topologies were proposed by Akram and Sitara [[7]- [8]], who also looked into the features that went along with them. Akram et al. investigated the residual product of fuzzy graph architectures [9]. Concepts of lexicographic-max product, fuzzy network structure, degree and total degree of a vertex in the lexicographic-max product in fuzzy graphs were introduced by Ali et al.[10]. Some of the maximum product's qualities and traits have been studied since Rao et al.[11] introduced the cubic fuzzy graph structure. Through the introduction of a vertex's degree and total degree in the product of a maximum of two cubic fuzzy graph structures, its computation techniques were classified according to various criteria on the vertex and edge membership functions. In Krishna and Radha [12], the difference mean fuzzy graph and the difference mean edge were presented. This work found the necessary conditions for an edge to be a difference mean edge in the direct sum of two fuzzy graphs. Furthermore, it is deduced that a difference mean fuzzy graph was produced by the direct sum of two fuzzy graphs.

1.2. Intuitionistic Fuzzy Graphs

As an extension of fuzzy sets, Atanassov [13] developed the idea of an $\mathscr{IFS}$. To the fuzzy set concept, Atanassov added extra elements that establish the degree of non-membership. While $\mathscr{IFS}$s provide both the degree of membership and the degree of non-membership, which are essentially independent of one another, fuzzy sets simply provide the degree of membership; the sole condition is that the total of these two degrees cannot be more than one. Numerous disciplines, including computer science, engineering, mathematics, medicine, chemistry, and economics, have used $\mathscr{IFS}$s. The topic of intuitionistic fuzzy interpretations of multi-person, multi-criteria decision making was covered by Atanassov [14]. $\mathscr{IFG}$s were first introduced by Parvathi and Karunambigai [15].The authors discussed a few $\mathscr{IFG}$ characteristics and presented a number of different concepts. These ideas were examined with appropriate visual aids.

On $\mathscr{IFG}$s, Sahoo and Pal [16] established three operations: direct product, semi-strong product, and strong product. They also looked at a number of intriguing outcomes related to the surgeries. Furthermore, it is shown that all strong $\mathscr{IFG}$'s products are also strong $\mathscr{IFG}$s. Ultimately, product $\mathscr{IFG}$s were defined and a number of intriguing findings were examined. Several novel concepts on extended architectures and their applications in various kinds of $\mathscr{IFG}$s were presented by Akram [17]. The concept of intuitionistic fuzzy incidence graphs ($\mathscr{IFIG}$s) and some of its characteristics were introduced by Nazeer [18]. A variety of operations in an $\mathscr{IFIG}$ were also examined, such as the composition, tensor product, normal product, and cartesian product (CP). The technique to calculate the degree of $\mathscr{IFIG}$s acquired by the use of composition, tensor product, normal product, and CP was explained.

The vertex degree of the cartesian product of an intuitionistic fuzzy network was examined by Pathinathan and Rosline [19]. Sahoo et al. [20] developed these concepts further to address issues in intuitionistic fuzzy situations. A novel definition of strong arc was employed by Shao [22] in relation to the degree of connectedness in $\mathscr{IFG}$s.

In order to give concise explanations of operations on directed rough fuzzy graphs and their effects on vertex degrees with examples, Nawaz and Ahmad [23] studied and investigated the degree of vertices in directed rough fuzzy graphs created by particular operations. Complex t-$\mathscr{IFG}$s (CTIFGs) are a valuable tool for studying and illustrating linkages across many applications that are challenging to recognise. Kaviyarasu et al.'s work from 2024 [24] outlined the concept of CTIFGs. The book also shows how CTIFGs, when evaluating a physical context, may form complicated linkages with several domains. Alqahtani [25] presented Hesitant bipolar-valued $\mathscr{IFG}$s for social media group dominant identification in 2024. Along with positive and negative side membership and non-membership grades in each element of a network, the suggested technique offers a more comprehensive framework for evaluating the most dominating and self-persistent individual in a social network across multi-level qualities. Lexical products of two $\mathscr{IFG}$s were examined by Syamala and Balasubramanian [22]. These products are known as the lexicographic min product and lexicographic max product, and they are defined similarly to the idea of lexicographic product in crisp graph theory.

This study defines specific notations of $\mathscr{IFG}$ structures by applying the notion of $\mathscr{IFS}$s to a graph structure. $\mathscr{K}$-Lexiographic-max product, degree and total degree of a vertex in $\mathscr{K}$-Lexicographic-max product, and $\mathscr{IFG}$ structure are introduced. Additionally, several numerical examples are provided to clarify the suggested principles. Specifically, intuitionistic fuzzy network topologies are applied in decision-making processes with the purpose of detecting fraud in online transactions.

1.3. Motivation

E-commerce platforms like Amazon, PayPal, Alibaba and many others created an unlimited number of possible transactions and all sorts of them are favorite targets for frauds. The method of detecting and avoiding fraud in such platforms is not a simple task, it has to use sophisticated methods that can function with uncertainty, hesitation or even with complexity belonging to the specific cases of online fraud. These coupled problems pose significant difficulties which are not well solved by the methods of graph theory in the classical sense. The rationale for this research arises from the desire to implement more flexible and robust mathematical structures into the current strategies applied to detect fraud incidences. $\mathscr{IFG}$ seem to be a promising direction in expressing uncertainty in decision-making. This paper focuses on the use of $\mathscr{IFG}$s in identifying fraud incidences during online transactions before offering a more sophisticated solution to the challenges inherent in such work.

1.4. Novelty

The innovation of this study is departed from using intuitionistic fuzzy set on graphs to develop an enhanced model for detecting fraudsters in e-commerce webpage. Unlike the standardized graph models, there is a level of uncertainty hesitation included in the Independent fraud graphs which is more realistic as it would be in real-life fraud investigations. The expansion of the new $\mathscr{IFG}$ structure based on $\mathscr{K}$-lexicographic-max product and introduction of new notations of vertex degrees enriches the perspective of computational intelligence. Given that $\mathscr{IFG}$s incorporation features numerous numerical examples and decision making algorithms, this paper provides a direct method of applying $\mathscr{IFG}$s methods and systematically trying them in actual cases of fraud detection about which most of the large e-commerce and financial platforms advertise. When used in this manner, this paradigm provides a new angle for managing the epistemological and ontological difficulties inherent in the classifications of online fraud.

1.5. The article's Structure

In section 1 and 2 the introduction and basic concepts of $\mathscr{IFG}$s and their properties were given. In section 3 the idea of $\mathscr{K}$-Lexicographic-Max product of $\mathscr{IFG}_s$ and their properties were discussed. The application of detection of fraud in online transactions using $\mathscr{IFG}_s$ were analyzed in section 4 and Comparison Analysis given in Section 5. Finally the conclusion discussed in Section 6.

2. Preliminaries

Definition 1 [3] A fuzzy graph $\mathscr{G} = (\mathscr{V},E)$ of functions $\mu:\mathscr{V} \to [0,1]$ and $\eta:E \to [0,1]$ such that for all $u,v \in \mathscr{V}$. Then the edge membership is define by $\eta(u,v) \le \min\{\mu(u),\mu(v)\}$. In this fuzzy set, the vertex set is denoted as μ and the edge set is denoted as η.

Definition 2 [3] A fuzzy graph $\mathscr{G}^* = (\mathscr{V},E^*)$ is said to be an complete

$$\eta(u,v) = \min\{\mu(u),\mu(v)\}, \forall u,v \in \mu *.$$

Definition 3 [37] A fuzzy graph $\mathscr{G}$ is said to be a effective edge if

$$\eta(u,v) = \min\{\mu(u),\mu(v)\} \forall u,v \in \eta *.$$

Definition 4 [16] An $\mathscr{IFG}$ is of the form $\mathscr{G} = (\mathscr{V},E)$ where

1. $\mathscr{V} = \{u_1,u_2,u_3,.....,u_n\}$, such that $\mu_\alpha:\mathscr{V} \to [0,1]$ and $\mu_\beta:\mathscr{V} \to [0,1]$ denote the degree of membership and non-membership of the vertices $u_i \in \mathscr{V}$ respectively, such that $0 \le \mu_\alpha(u_i) + \mu_\beta(u_i) \le 1, \forall u_i \in \mathscr{V} (i = 0,1,2,.....n)$.
2. $\eta_\alpha:\mathscr{V} \times \mathscr{V} \to [0,1]$ and $\eta_\beta:\mathscr{V} \times \mathscr{V} \to [0,1]$, where $\eta_\alpha(u_i u_j)$ and $\eta_\beta(u_i u_j)$ denote the degree of membership and non-membership of the edge $(u_i u_j) \in E$, respectively, such that

$$\eta_\alpha(u,v) \le \min\{\mu_\alpha(u_i),\mu_\alpha(u_j)\},\ \eta_\beta(u_i,u_j) \le \max\{\mu_\beta(u_i),\mu_\beta(u_j)\} \text{ and}$$
$$0 \le \eta_\alpha(u_i u_j) + \eta_\beta(u_i u_j) \le 1, \text{ for edge } (u_i u_j) \in E.$$

Definition 5 [16] Let $G = ((\mu_\alpha,\mu_\beta),(\eta_\alpha,\eta_\beta))$ be an $\mathscr{IFG}$ the order of G is defined to be $O(\mathscr{G}) = (O_{\mu_\alpha}(\mathscr{G}),O_{\mu_\beta}(\mathscr{G}))$, where $O_{\mu_\alpha}(\mathscr{G}) = \sum_{u \in \mathscr{V}} \eta_\alpha(u_i u_j)$ and $O_{\mu_\beta}(\mathscr{G}) = \sum_{u \in \mathscr{V}} \eta_\beta(u_i u_j)$.

Definition 6 [38] Let $\mathscr{G}_1 = ((\mu_\alpha,\mu_\beta),(\eta_\alpha,\eta_\beta))$ and $\mathscr{G}_2 = ((\mu'_\alpha,\mu'_\beta),(\eta'_\alpha,\eta'_\beta))$ be two $\mathscr{IFG}$s. The max product of two $\mathscr{IFG}$ $\mathscr{G}_1$ and G_2 is denoted by $\mathscr{G}_1 \times_{max} \mathscr{G}_2 = (\mathscr{V}_1 \times_{max} \mathscr{V}_2, E_1 \times_{max} E_2), E_1 \times_{max} E_2 = \{(u_1,v_1)(u_2,u_2) | u_1 = u_2, v_1 v_2 \in E_2 \text{ or } v_1 = v_2, u_1 u_2 \in E_1\}$ by $\mu_\alpha^{\mathscr{G}_1 \times_{max} \mathscr{G}_2}(u_1,v_1) = \max\{\mu_\alpha(u_1),\mu'_\alpha(v_1)\}$ and $\mu_\beta^{\mathscr{G}_1 \times_{max} \mathscr{G}_2}(u_1,v_1) = \min\{\mu_\beta(u_1),\mu'_\beta(v_1)\}$, for all $(u_1,v_1) \in \mathscr{V}_1 \times \mathscr{V}_2$ and

$$\eta_\alpha^{\mathscr{G}_1 \times_{max} \mathscr{G}_2}((u_1,v_1),(u_2,v_2)) = \begin{cases} \max\{\mu_\alpha(u_1),\eta'_\alpha(v_1 v_2)\} & \text{if} u_1 = u_2, v_1 v_2 \in E_2 \\ \max\{\eta_\alpha(u_1 u_2),\mu'_\alpha(v_1)\} & \text{if} v_1 = v_2, u_1 u_2 \in E_1 \end{cases}$$

$$\eta_{\beta}^{\mathscr{G}_1\times_{\max}\mathscr{G}_2}((u_1,v_1),(u_2,v_2)) = \begin{cases} \min\left\{\mu_\beta(u_1),\eta'_\beta(v_1v_2)\right\} & \text{if} u_1 = u_2, v_1v_2 \in E_2 \\ \min\left\{\eta_\beta(u_1u_2),\mu'_\beta(v_1)\right\} & \text{if} v_1 = v_2, u_1u_2 \in E_1 \end{cases}$$

3. $\mathscr{K}$-Lexicographic-Max product of $\mathscr{IFG}_s$

Definition 7 Let $\mathscr{G}_1 = \left\langle(\mu_\alpha,\mu_\beta),(\eta_\alpha,\eta_\beta)\right\rangle$ and $\mathscr{G}_2 = \left\langle(\mu'_\alpha,\mu'_\beta),(\eta'_\alpha,\eta'_\beta)\right\rangle$ denote two $\mathscr{IFG}_s$. Define $\mathscr{G}_1 = \left\langle(\mu_\alpha,\mu_\beta),(\eta_\alpha,\eta_\beta)\right\rangle$ with underlying crisp graph $G^*:(\mathscr{V},E)$ where $\mathscr{V} = \mathscr{V}_1 \times \mathscr{V}_2$,
$E = \{((u_1,v_1),(u_2,v_2))\ u_1u_2 \in E \text{ or } u_1 = u_2 \text{ and } v_1v_2 \in E \text{ or } v_1 = v_2\}$ by $\eta_\alpha(u_1,v_1) = \mu_\alpha(u_1) \wedge \mu_\alpha(v_1)$, $\eta_\beta(u_1,v_1) = \mu_\beta(u_1) \vee \mu_\beta(v_1)$ and $\eta^*_\alpha((u_1,v_1)(u_2,v_2)) = \eta^*_\alpha(u_1,u_2)$

Theorem. 1 If $\mathscr{G}_1 = \left\langle(\mu_\alpha,\eta_\alpha,\rho'_1,\rho'_2)\right\rangle$ and $\mathscr{G}_2 = \left\langle(\mu_\beta,\eta_\beta,\rho''_1,\rho''_2)\right\rangle$ are two effective $\mathscr{IFG}_s$ such that $\mu_\alpha \le \eta_\alpha$ and $\mu_\beta \le \eta_\beta$. If ρ'_i and ρ''_i are constant functions having similar values, then the $\mathscr{K}-\mathscr{LMP}$ of $\mathscr{G}_1$ and $\mathscr{G}_2$ is an effective $\mathscr{IFG}_s$.

Proof. Assume that $\mathscr{G}_1 = \left\langle(\mu_\alpha,\eta_\alpha,\rho'_1,\rho'_2)\right\rangle$ and $\mathscr{G}_2 = \left\langle(\mu_\beta,\eta_\beta,\rho''_1,\rho''_2)\right\rangle$ are two effective $\mathscr{IFG}_s$ such that $\mu_\alpha \le \eta_\alpha$ and $\mu_\beta \le \eta_\beta$. If ρ'_i and ρ''_i are constant functions with comparable values, then the $\mathscr{K}-\mathscr{LMP}$ definitions provides.

Case (i) $u_1u_2 \in E'_i$.

$$\begin{aligned}
\rho_i(u_1,v_1)(u_2,v_2) &= \rho'_i(u_1,u_2) \\
&= \rho''_i(v_1,v_2) \\
&= \min\{\mu_\alpha(v_1),\mu_\alpha(v_2)\} \\
&= \min\{\min\{\mu_\alpha(u_1),\mu_\alpha(v_1)\},\min\{\mu_\alpha(u_2),\mu_\alpha(v_2)\}\} \\
&= \min\{\eta_\alpha(u_1,v_1),\eta_\alpha(u_2,v_2)\} \text{ and} \\
\rho_i(u_1,v_1)(u_2,v_2) &= \rho'_i(u_1,u_2) \\
&= \rho''_i(v_1,v_2) \\
&= \max\left\{\mu_\beta(v_1),\mu_\beta(v_2)\right\} \\
&= \max\left\{\max\left\{\mu_\beta(u_1),\mu_\beta(v_1)\right\},\max\left\{\mu_\beta(u_2),\mu_\beta(v_2)\right\}\right\} \\
&= \max\left\{\eta_\beta(u_1,v_1),\eta_\beta(u_2,v_2)\right\}
\end{aligned}$$

case (ii) $u_1 = u_2, v_1v_2 \in E''_i$.

$$\begin{aligned}
\rho_i(u_1,v_1)(u_2,v_2) &= \min\left\{\mu_\alpha(u_1),\mu''_\alpha(v_1,v_2)\right\} \\
&= \eta_\alpha(v_1,v_2) \\
&= \min\{\mu_\alpha(v_1),\mu_\alpha(v_2)\} \\
&= \min\{\min\{\mu_\alpha(u_1),\mu_\alpha(v_1)\},\min\{\mu_\alpha(u_2),\mu_\alpha(v_2)\}\} \\
&= \min\{\eta_\alpha(u_1,v_1),\eta_\alpha(u_2,v_2)\}
\end{aligned}$$

and

$$\begin{aligned}
\rho_i(u_1,v_1)(u_2,v_2) &= \max\left\{\mu_\beta(u_1),\mu''_\beta(v_1,v_2)\right\} \\
&= \eta_\beta(v_1,v_2) \\
&= \max\left\{\mu_\beta(v_1),\mu_\beta(v_2)\right\} \\
&= \max\left\{\max\left\{\mu_\beta(u_1),\mu_\beta(v_1)\right\},\max\left\{\mu_\beta(u_2),\mu_\beta(v_2)\right\}\right\} \\
&= \max\left\{\eta_\beta(u_1,v_1),\eta_\beta(u_2,v_2)\right\}
\end{aligned}$$

Thus,

$$\rho_i(u_1,v_1)(u_2,v_2) = \min\{\mu_\alpha(u_1,v_1),\mu_\alpha(u_2,v_2)\}$$
$$\rho_i(u_1,v_1)(u_2,v_2) = \max\left\{\mu_\beta(u_1,v_1),\mu_\beta(u_2,v_2)\right\}$$

Example 1 Consider two $\mathscr{IFG}_s$ $\mathscr{G}_1 = \left\langle\mu_\alpha,\eta_\alpha,\rho'_1,\rho'_2\right\rangle$ and $\mathscr{G}_2 = \left\langle\mu_\beta,\eta_\beta,\rho''_1,\rho''_2,\rho''_3,\rho''_4\right\rangle$ with underlying crisp Graphs $\mathscr{G}^*_1 = \left\langle\mathscr{V}_1,R'_1,R'_2\right\rangle$ and $\mathscr{G}^*_2 = \left\langle\mathscr{V}_2,R''_1,R''_2,R''_3,R''_4\right\rangle$, respectively, which are shown in Figure 1, where $R'_1 = \{u_1u_2\}, R'_2 = \{u_3u_4\}, R''_1 = \{v_1v_2\}, R''_1 = \{v_1v_3\}, R''_3 = \{v_3v_4\}, R'_4 = \{v_4v_1\}$. The $\mathscr{K}-\mathscr{LMP}$ of the above $\mathscr{IFG}_s$ $\mathscr{G}_1 = \left\langle\mu_\alpha,\eta_\alpha,\rho'_1,\rho'_2\right\rangle$ and $\mathscr{G}_2 = \left\langle\mu_\beta,\eta_\beta,\rho''_1,\rho''_2,\rho''_3,\rho''_4\right\rangle$ is shown in Figure 2. In the $\mathscr{K}-\mathscr{LMP}$, ρ'_i and ρ''_i edges belong to the ρ_i $\mathscr{IFS}$s.

Figure 1: *IFG*

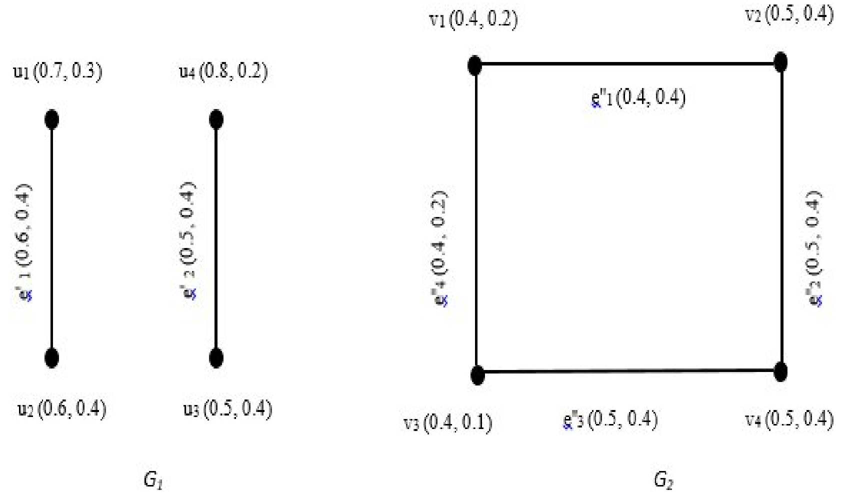
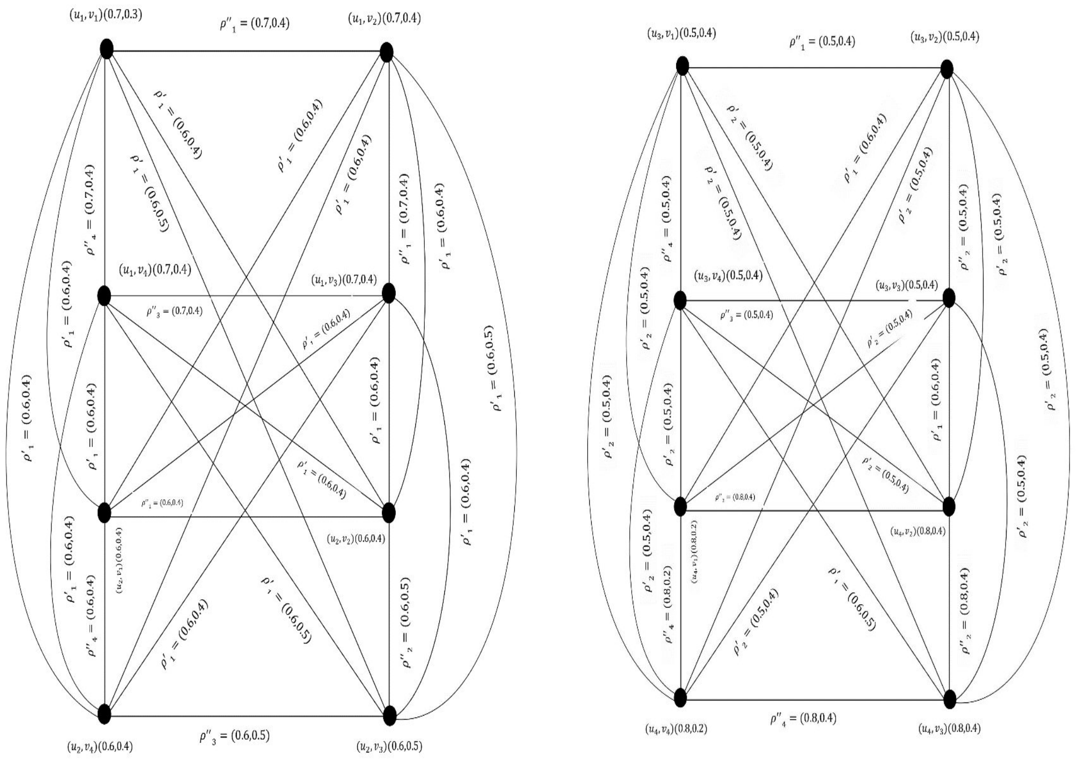


Figure 2: $\mathscr{K}-\mathscr{LMP}$ **of IFGs**



Example 2 Consider two $\mathscr{IFG}$s $\mathscr{G}_1 = \langle \mu_\alpha, \eta_\alpha, \rho'_1, \rho'_2 \rangle$ and $\mathscr{G}_2 = \langle \mu_\beta, \eta_\beta, \rho''_1, \rho''_2, \rho''_3, \rho''_4 \rangle$ as shown in Figure 3.

Table 1: Membership and Non-Membership of value of Fig 1

$\rho'_i(v_i,v_j)$	Membership value	Non-Membership of value
$\rho'_1(u_1,u_2)$	0.6	0.4
$\rho'_2(u_3,u_4)$	0.5	0.4
$\rho''_i(u_i,u_j)$	Membership value	Non-Membership of value
$\rho''_2(v_1,v_2)$	0.4	0.4
$\rho''_2(v_2,v_4)$	0.5	0.4
$\rho''_2(v_3,v_4)$	0.5	0.4
$\rho''_2(v_3,v_1)$	0.4	0.2

Hence,$\mathscr{G}_1 = \langle \mu_\alpha,\eta_\alpha,\rho'_1,\rho'_2 \rangle$ and $\mathscr{G}_2 = \langle \mu_\beta,\eta_\beta,\rho''_1,\rho''_2,\rho''_3,\rho''_4 \rangle$ are effective $\mathscr{IFGS}$ more over $\mu_\alpha = \eta_\alpha$, $\mu_\beta = \eta_\beta$, ρ'_i and $\rho''_i, i = 1,2,3,4$ are constant functions having the same value. ie $\langle 0.5,0.2 \rangle$. The $\mathscr{K}-\mathscr{LMP}$ of $\mathscr{G}_1 = \langle \mu_\alpha,\eta_\alpha,\rho'_1,\rho'_2 \rangle$ and $\mathscr{G}_2 = \langle \mu_\beta,\eta_\beta,\rho''_1,\rho''_2,\rho''_3,\rho''_4 \rangle$ is shown Figure 4. It is clear from Figure 4, that

Figure 3: IFGS

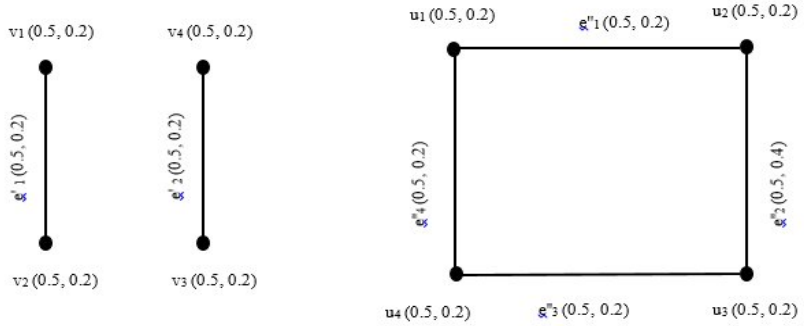


Figure 4: $\mathscr{K}-\mathscr{LMP}$ of IFGS

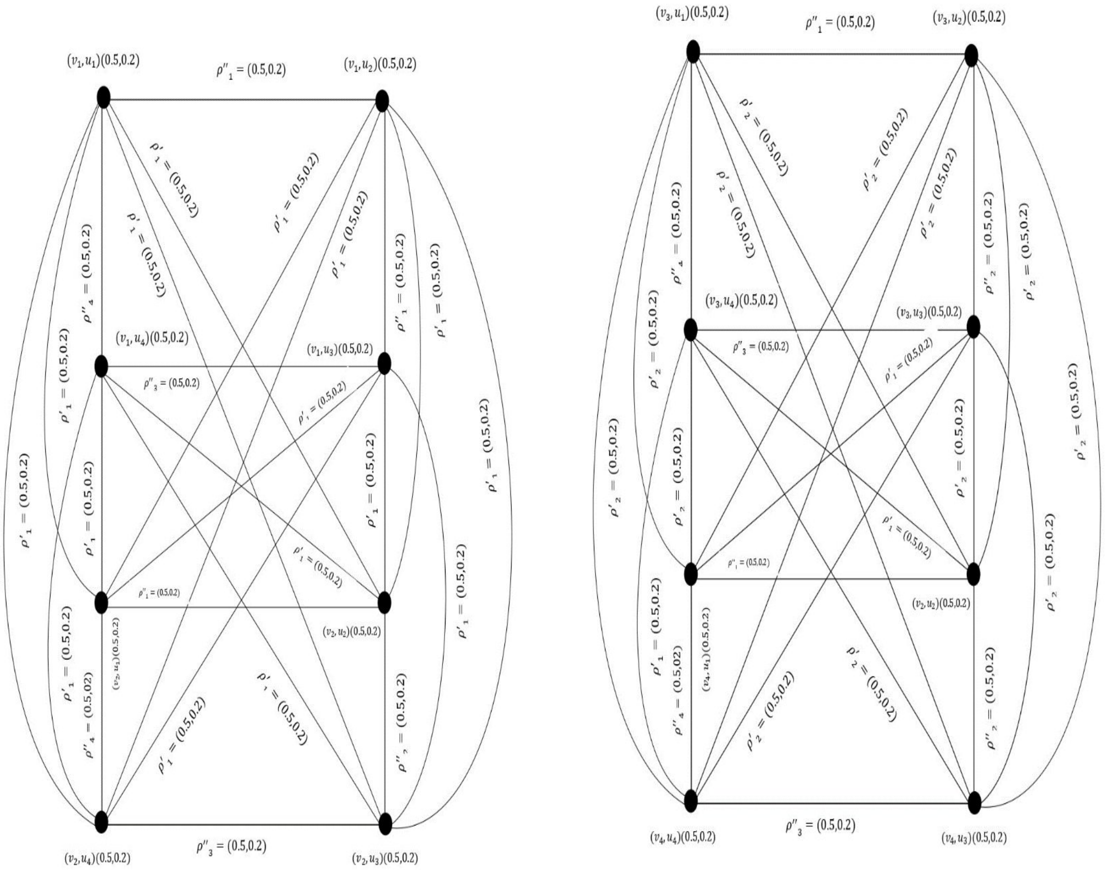


Table 2: Membership and Non-Membership of value of Fig 4

Table 2: Membership and Non-Membership of value of Fig

Definition 8 In the $\mathscr{K}-\mathscr{LMP}\ \mathscr{G}_1[\mathscr{G}_2]^{max}=\langle\mu,\eta,\rho_1,\rho_2,.....\rho_n\rangle$ of $\mathscr{IFG}_s\ \mathscr{G}_1=\left\langle\mu_\alpha,\eta_\alpha,\rho_1',\rho_2',....\rho_n'\right\rangle$ and $\mathscr{G}_2=\left\langle\mu_\beta,\eta_\beta,\rho_1'',\rho_2'',.....,\rho_n''\right\rangle$ the degree of a vertex is defined as

$$\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_i,u_j)=\Sigma_{v_iv_k\in\mathscr{E}_i'u_l\in V_2}\alpha_i'(v_iv_k)+\Sigma_{u_ju_l\in\mathscr{E}_j'',v_i=v_k}min\left\{\alpha_j''(u_ju_l),\alpha_l(v_i)\right\}$$

$$\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_i,v_j)=\Sigma_{v_iv_k\in\mathscr{E}_i'u_l\in V_2}\beta_i'(v_iv_k)+\Sigma_{u_ju_l\in\mathscr{E}_j'',v_i=v_k}max\left\{\beta_j''(u_ju_l),\eta(v_i)\right\}$$

α_i and β_i be the degree of a vertex in the $\mathscr{K}-\mathscr{LMP}\ \mathscr{G}_1[\mathscr{G}_2]^{max}$ is defined as

$$\alpha_i-\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_i,u_j)=\Sigma_{v_iv_k\in\mathscr{E}_i'u_l\in V_2}\alpha_i'(v_i,v_k)+\Sigma_{u_ju_l\in\mathscr{E}_j'',v_i=v_k}min\left\{\alpha_i''(u_j,u_l),\alpha_l(v_i)\right\}$$

$$\beta_i-\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_i,u_j)=\Sigma_{v_iv_k\in\mathscr{E}_i'u_l\in V_2}\beta_i'(v_i,v_k)+\Sigma_{u_ju_l\in\mathscr{E}_j'',v_i=v_k}max\left\{\beta_i''(u_j,u_l),\beta_l(v_i)\right\}$$

Example 3 Consider two $\mathscr{IFG}$s $\mathscr{G}_1=\left\langle(\mu_\alpha,\eta_\alpha,\rho_1',\rho_2')\right\rangle$ and $\mathscr{G}_2=\left\langle(\mu_\beta,\eta_\beta,\rho_1'',\rho_2'')\right\rangle$ which are shown in Figure 5. Figure 6 displays the $\mathscr{K}-\mathscr{LMP}$ of $\mathscr{IFG}_s\ \mathscr{G}_1$ and $\mathscr{G}_2$. The following formula is used to calculate the degree of all vertices in the $\mathscr{K}-\mathscr{LMP}$.

Figure 5: IFGS

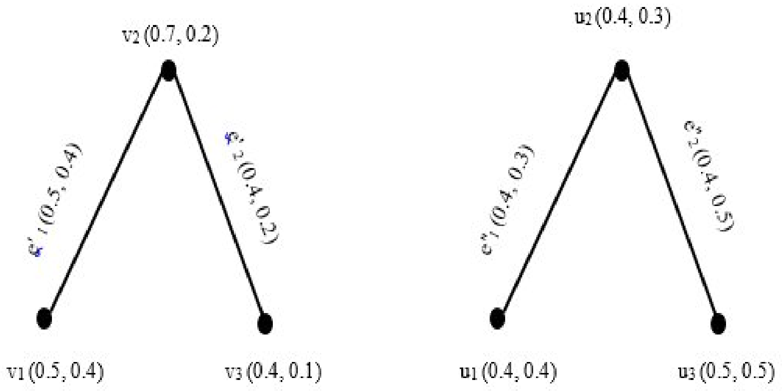
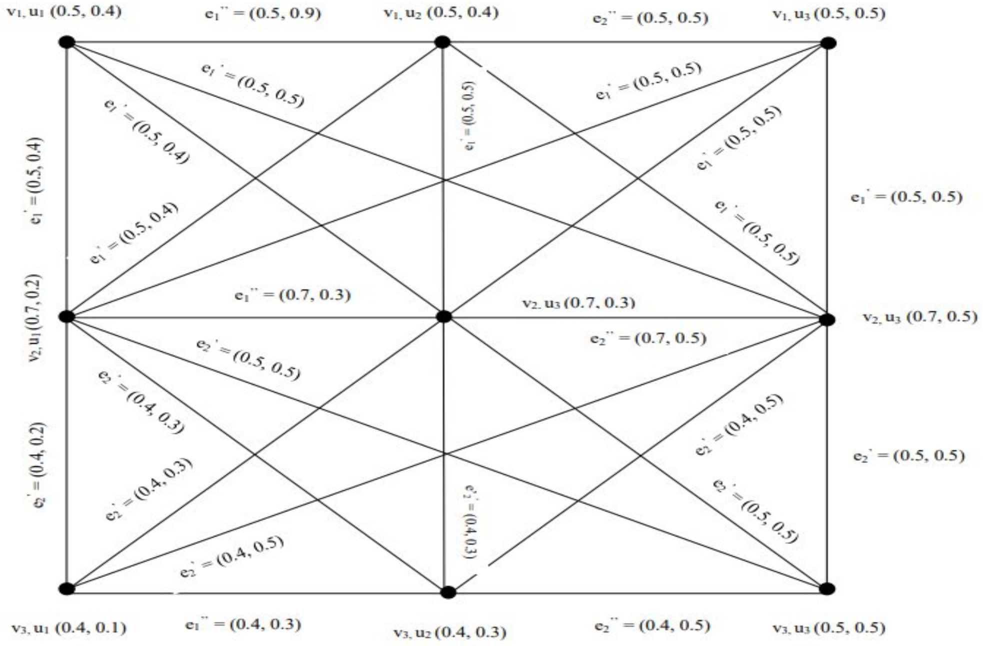


Figure 6: $\mathscr{K}-\mathscr{LMP}$ of IFGS



$$\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_i,u_j) = \Sigma_{v_iv_k\in\mathscr{E}_i' u_l\in\mathscr{V}_2}\alpha_i'(v_iv_k) + \Sigma_{u_ju_l\in\mathscr{E}_j'',v_i=v_k} min\left\{\alpha_j''(u_ju_l),\alpha_l(v_i)\right\}$$
$$\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_i,v_j) = \Sigma_{v_iv_k\in\mathscr{E}_i' u_l\in\mathscr{V}_2}\beta_i'(v_iv_k) + \Sigma_{u_ju_l\in\mathscr{E}_j'',v_i=v_k} max\left\{\beta_j''(u_ju_l),\beta_l(v_i)\right\}$$

Table 3: Degree of all vertices in the $\mathscr{K}-\mathscr{LMP}$

$\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_i,u_j)$	**Membership of value**	**Non-Membership of value**
(v_1,u_1)	1.9	1.7
(v_1,u_2)	2.3	2.1
(v_1,u_3)	1.9	1.7
(v_2,u_1)	3.1	2.1
(v_2,u_2)	3.5	2.6
(v_2,u_3)	3.1	2.3
(v_3,u_1)	1.6	0.9
(v_3,u_2)	2.0	1.4
(v_3,u_3)	1.6	1.1

α_i and β_i be the degree of a vertex in the $\mathscr{K}-\mathscr{LMP}\ \mathscr{G}_1[\mathscr{G}_2]^{max}$ is defined as

$$\alpha_i - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_i,u_j) = \Sigma_{v_iv_k\in\mathscr{E}_i' u_l\in V_2}\alpha_i'(v_i,v_k) + \Sigma_{u_ju_l\in\mathscr{E}_j'',v_i=v_k} min\left\{\alpha_i''(u_j,u_l),\alpha_l(v_i)\right\}$$
$$\beta_i - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_i,u_j) = \Sigma_{v_iv_k\in\mathscr{E}_i' u_l\in V_2}\beta_i'(v_i,v_k) + \Sigma_{u_ju_l\in\mathscr{E}_j'',v_i=v_k} max\left\{\beta_i''(u_j,u_l),\beta_l(v_i)\right\}$$

Table 4: α_i and β_i be the degree of a vertex in the $\mathscr{K}-\mathscr{LMP}$

Vertices	$\alpha_i - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_i,u_j)$	$\beta_i - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_i,v_j)$
$\mu_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_1,u_1)$	2.0	1.7
$\mu_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_1,u_2)$	1.9	1.6
$\mu_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_1,u_3)$	1.5	1.2
$\mu_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_2,u_1)$	1.9	1.5
$\eta_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_2,u_1)$	1.2	0.6
$\mu_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_2,u_2)$	1.9	0.5
$\eta_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_2,u_2)$	1.6	1.4
$\mu_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_2,u_3)$	1.5	1.2
$\eta_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_2,u_3)$	1.6	1.1
$\mu_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_3,u_1)$	0.4	0.3
$\eta_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_3,u_1)$	1.2	0.6
$\mu_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_3,u_2)$	0.4	0.3
$\eta_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_3,u_2)$	1.6	1.1
$\eta_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_3,u_3)$	1.6	1.1
$\eta_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_1,u_2)$	0.4	0.5
$\eta_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_1,u_3)$	0.4	0.5

Theorem 2 If $\mathscr{G}_1$ and $\mathscr{G}_2$ are two $\mathscr{IFG}_\mathscr{J}$ such that $\eta_\alpha'' \geq \mu_\alpha$, $\eta_\beta'' \leq \mu_\beta$, then $\mathscr{IFG}_\mathscr{J}$ $\mathscr{G}_1$ and $\mathscr{G}_2$ have a $\mathscr{K}-\mathscr{LMP}\ \mathscr{G}_1[\mathscr{G}_2]^{max}$, where the degree of the vertex is provided by $\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_i,u_j) = |V_2|\ \mathfrak{d}_{\mathscr{G}_1}(v_i) + \mathfrak{d}_{\mathscr{G}_2}(u_j)$.

Proof. Let $\mathscr{G}_1$ and $\mathscr{G}_2$ are two $\mathscr{IFG}_s$ such that $\eta_\alpha'' \geq \mu_\alpha, \eta_\beta'' \leq \mu_\beta$. This implies that $\eta_\alpha'' \wedge \mu_\alpha = \eta_\alpha''$ and $\eta_\beta'' \vee \mu_\beta = \eta_\beta''$. Then the degree of any vertex $(v_i,u_j) \in \mathscr{V}_1 \times \mathscr{V}_2$ is given by $\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_i,u_j) = \Sigma_{v_iu_j \in \mathscr{E}_1 u_j \in V_2} \eta_\alpha'(v_i,u_j) + \Sigma_{u_ju_l \in \mathscr{E}_2, v_i = v_k} min\left\{\eta_\alpha''(u_j,u_l), \mu_\alpha(v_i)\right\}$

$$= |\mathscr{V}_2| \Sigma_{v_iu_j \in \mathscr{E}_1} \eta_\alpha'(v_i,u_j) + \Sigma_{u_ju_l \in \mathscr{E}_2, v_i = v_k} min\left\{\eta_\alpha''(u_j,u_l), \mu_\alpha(v_i)\right\}$$

$$= |\mathscr{V}_2|\, \mathfrak{d}_{\mathscr{G}_1}(v_i) + \mathfrak{d}_{\mathscr{G}_2}\left(u_j\right) \text{ and}$$

$$\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_i,u_j) = \Sigma_{v_iu_j \in \mathscr{E}_1 u_j \in V_2} \eta_\beta'(v_i,u_j) + \Sigma_{u_ju_l \in \mathscr{E}_2, v_i = v_k} max\left\{\eta_\beta''(u_j,u_l), \mu_\beta(v_i)\right\}$$

$$= |\mathscr{V}_2| \Sigma_{v_iu_j \in \mathscr{E}_1} \eta_\beta'(v_i,u_j) + \Sigma_{u_ju_l \in \mathscr{E}_2, v_i = v_k} max\left\{\eta_\beta''(u_j,u_l), \mu_\beta(v_i)\right\}$$

$$= |\mathscr{V}_2| \mathfrak{d}_{\mathscr{G}_1}(v_i) + \mathfrak{d}_{\mathscr{G}_2}(u_j)$$

Theorem 3 If $\mathscr{G}_1$ and $\mathscr{G}_2$ are two $\mathscr{IFG}_s$ such that $\eta_\alpha'' \leq \mu_\alpha, \eta_\beta'' \leq \mu_\beta$, then the α_i and β_i-membership and non-membership degree of the vertex in the $\mathscr{LMP}$ $\mathscr{G}_1[\mathscr{G}_2]^{max}$ of $\mathscr{IFG}_s$ $\mathscr{G}_1$ and $\mathscr{G}_2$ is given by

$$\alpha_i - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_i,u_j) = |\mathscr{V}_2| \left[\alpha_i' - \mathfrak{d}_{\mathscr{G}_1}(v_i)\right] + \left[\alpha_i'' - \mathfrak{d}_{\mathscr{G}_2}(u_j)\right]$$

$$\beta_i - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_i,u_j) = |\mathscr{V}_2| \left[\beta_i' - \mathfrak{d}_{\mathscr{G}_1}(v_i)\right] + \left[\beta_i'' - \mathfrak{d}_{\mathscr{G}_2}(u_j)\right]$$

Proof. Let $\mathscr{G}_1$ and $\mathscr{G}_2$ are two $\mathscr{IFG}_s$ such that $\eta_\alpha'' \geq \mu_\alpha, \eta_\beta'' \leq \mu_\beta$. This implies that $\eta_\alpha'' \wedge \mu_\alpha = \eta_\alpha''$ and $\eta_\beta'' \vee \mu_\beta = \eta_\beta''$. then the α_i and β_i-membership and non-membership degree of the vertex in the $\mathscr{LMP}$ $\mathscr{G}_1[\mathscr{G}_2]^{max}$ is given by

$$\alpha_i - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_i,u_j) = \Sigma_{v_iu_j \in \mathscr{E}_1 u_j \in V_2} \eta_\alpha'(v_i,u_j) + \Sigma_{u_ju_l \in \mathscr{E}_2, v_i = v_k} min\left\{\eta_\alpha''(u_j,u_l), \mu_\alpha(v_i)\right\}$$

$$= |\mathscr{V}_2| \Sigma_{v_iu_j \in \mathscr{E}_1} \eta_\alpha'(v_i,u_j) + \Sigma_{u_ju_l \in \mathscr{E}_2, v_i = v_k} min\left\{\eta_\alpha''(u_j,u_l), \mu_\alpha(v_i)\right\}$$

$$= |\mathscr{V}_2| \left[\alpha_i' - \mathfrak{d}_{\mathscr{G}_1}(v_i)\right] + \left[\alpha_i'' - \mathfrak{d}_{\mathscr{G}_2}\left(u_j\right)\right] \text{ and}$$

$$\beta_i - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_i,u_j) = \Sigma_{v_iu_j \in \mathscr{E}_1 u_j \in \mathscr{V}_2} \eta_\beta'(v_i,u_j) + \Sigma_{u_ju_l \in \mathscr{E}_2, v_i = v_k} max\left\{\eta_\beta''(u_j,u_l), \mu_\beta(v_i)\right\}$$

$$= |\mathscr{V}_2| \Sigma_{v_iu_j \in \mathscr{E}_1} \eta_\beta'(v_i,u_j) + \Sigma_{u_ju_l \in \mathscr{E}_2, v_i = v_k} max\left\{\eta_\beta''(u_j,u_l), \mu_\beta(v_i)\right\}$$

$$= |\mathscr{V}_2| \left[\beta_i' - \mathfrak{d}_{\mathscr{G}_1}(v_i)\right] + \left[\beta_i'' - \mathfrak{d}_{\mathscr{G}_2}(u_j)\right]$$

Example 4 Consider two $\mathscr{IFG}_s$ $\mathscr{G}_1$ and $\mathscr{G}_2$ shown in Figure 7. In the graph $\eta_\alpha'' \geq \mu_\alpha, \eta_\beta'' \leq \mu_\beta$. Figure 8 displays the $\mathscr{LMP}\mathscr{G}_1$ and $\mathscr{G}_2$. The following formula is implied by theorem3 to as certain the degree of vertices in $\mathscr{K} - \mathscr{LMP}$.

Figure 7: IFGS

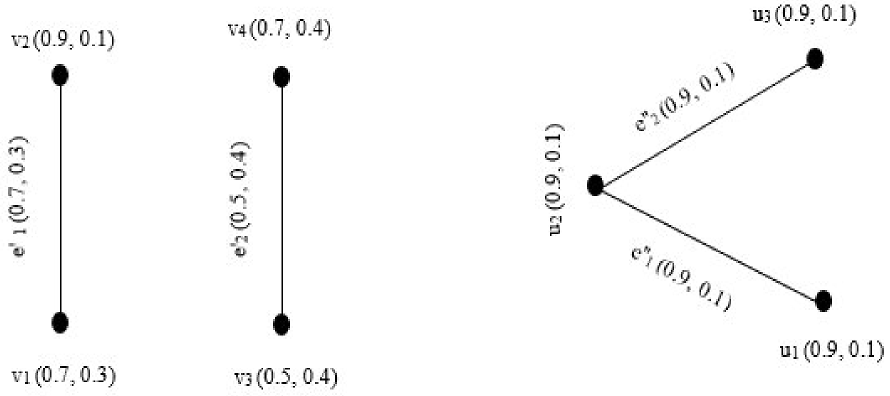
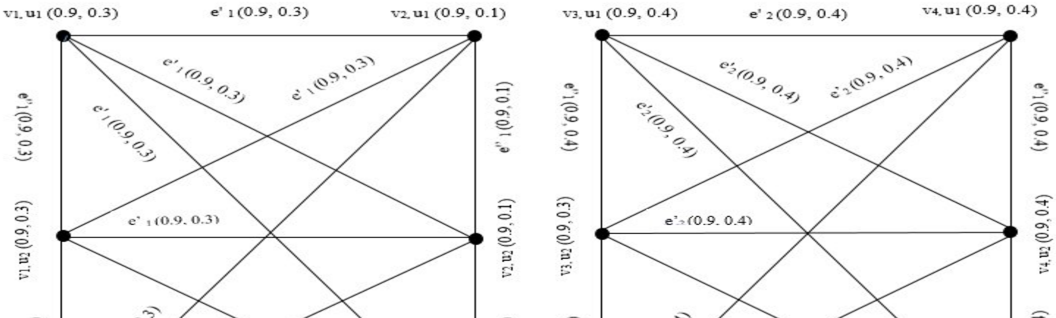


Figure 8: $\mathscr{K} - \mathscr{LMP}$ of IFGS



$$\alpha_i - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_i,u_j) = |\mathscr{V}_2| \left[\alpha_i' - \mathfrak{d}_{\mathscr{G}_1}(v_i)\right] + \left[\alpha_i'' - \mathfrak{d}_{\mathscr{G}_2}(u_j)\right]$$
$$\beta_i - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_i,u_j) = |\mathscr{V}_2| \left[\beta_i' - \mathfrak{d}_{\mathscr{G}_1}(v_i)\right] + \left[\beta_i'' - \mathfrak{d}_{\mathscr{G}_2}(u_j)\right]$$

Table 5: α_i and β_i be the degree of a vertex in the $\mathscr{K} - \mathscr{LMP}$

Vertices	$\alpha_i - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_1,u_1)$	$\beta_i - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_i,v_j)$
$\mu_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_1,u_1)$	3.6	1.2
$\mu_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_1,u_2)$	3.6	1.2
$\eta_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_1,u_2)$	0.9	0.3
$\mu_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_1,u_3)$	2.7	0.9
$\eta_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_1,u_3)$	0.9	0.3
$\mu_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_2,u_1)$	3.6	1
$\eta_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_2,u_1)$	0	0
$\mu_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_2,u_2)$	3.6	1
$\eta_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_2,u_2)$	0.9	0.1
$\mu_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_2,u_3)$	2.7	0.4
$\eta_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_2,u_3)$	0.9	0.1
$\mu_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_3,u_1)$	0.9	0.4
$\eta_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_3,u_1)$	2.7	1.2
$\mu_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_3,u_2)$	0.9	0.4
$\eta_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_3,u_2)$	3.6	1.6
$\mu_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_3,u_3)$	0	0
$\eta_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_3,u_3)$	3.6	1.6
$\mu_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_4,u_1)$	0.9	0.4
$\eta_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_4,u_1)$	2.7	1.2
$\mu_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_4,u_2)$	0.9	0.4
$\eta_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_4,u_2)$	3.6	1.6
$\mu_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_4,u_3)$	0.9	0.4
$\eta_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_4,u_3)$	3.6	1.6

Theorem 4 If the $\mathscr{G}_1$ and $\mathscr{G}_2$ are two $\mathscr{IFG}_{\mathscr{J}}$ such that $\eta_\alpha'' \le \mu_\alpha$, $\eta_\beta'' \le \mu_\beta$, then the $\mathscr{K} - \mathscr{LMP}$ $\mathscr{G}_1[\mathscr{G}_2]^{max}$ of $\mathscr{IFG}_{\mathscr{J}}$, $\mathscr{G}_1$ and $\mathscr{G}_2$ yields the α_i and β_i-membership and non-membership degree of the vertex as follows;

$$\alpha_i - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_i,u_j) = |\mathscr{V}_2| \left[\alpha_i' - \mathfrak{d}_{\mathscr{G}_1}(v_i)\right] + \left[\mathfrak{d}_{\mathscr{G}_2}(u_j)\mu_\alpha(v_i)\right]$$
$$\beta_i - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_i,u_j) = |\mathscr{V}_2| \left[\beta_i' - \mathfrak{d}_{\mathscr{G}_1}(v_i)\right] + \left[\mathfrak{d}_{\mathscr{G}_2}(u_j)\mu_\beta(v_i)\right]$$

Proof. Let $\mathscr{G}_1$ and $\mathscr{G}_2$ are two $\mathscr{IFG}_{\mathscr{J}}$ such that $\eta_\alpha'' \le \mu_\alpha$, $\eta_\beta'' \le \mu_\beta$. This implies that $\eta_\alpha'' \wedge \mu_\alpha = \eta_\alpha''$ and $\eta_\beta'' \vee \mu_\beta = \eta_\beta''$. Then the α_i and β_i-membership and non-membership degree of the vertex in the $\mathscr{LMP}$ $\mathscr{G}_1[\mathscr{G}_2]^{max}$ is given by

$$\begin{aligned}\alpha_i - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_i,u_j) &= \Sigma_{v_iu_j \in \mathscr{E}_1 u_j \in V_2} \eta_\alpha'(v_i,u_j) + \Sigma_{u_ju_l \in \mathscr{E}_2, v_i = v_k} min\left\{\eta_\alpha''(u_j,u_l), \mu_\alpha(v_i)\right\} \\ &= |\mathscr{V}_2| \Sigma_{v_iu_j \in \mathscr{E}_1} \eta_\alpha'(v_i,u_j) + \Sigma_{u_ju_l \in \mathscr{E}_2, v_i = v_k} \mu_\alpha(v_i)\end{aligned}$$

$$\alpha_i - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}\left(v_i,u_j\right) = |\mathscr{V}_2|\left[\alpha_i' - \mathfrak{d}_{\mathscr{G}_1}(v_i)\right] + \left[\mathfrak{d}_{\mathscr{G}_2}\left(u_j\right)\mu_\alpha(v_i)\right] \text{ and}$$
$$\beta_i - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_i,u_j) = \Sigma_{v_iu_j\in\mathscr{E}_1 u_j\in V_2}\eta_\beta'(v_i,u_j) + \Sigma_{u_ju_l\in\mathscr{E}_2,v_i=v_k} max\left\{\eta_\beta''(u_j,u_l),\mu_\beta(v_i)\right\}$$
$$= |\mathscr{V}_2|\Sigma_{v_iu_j\in\mathscr{E}_1}\eta_\beta'(v_i,u_j) + \Sigma_{u_ju_l\in\mathscr{E}_2,v_i=v_k}\mu_\beta(v_i)$$
$$\beta_i - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_i,u_j) = |\mathscr{V}_2|\left[\beta_i' - \mathfrak{d}_{\mathscr{G}_1}(v_i)\right] + \left[\mathfrak{d}_{\mathscr{G}_2}(u_j)\mu_\alpha(v_i)\right]$$

Corollary 1 If the $\mathscr{G}_1$ and $\mathscr{G}_2$ are two $\mathscr{IFG}_{\mathscr{J}}$ such that $\eta_\alpha'' \leq \mu_\alpha$, $\eta_\beta'' \leq \mu_\beta$, also μ_α and μ_β are considered as a constant functions having the value C_1 and C_2 respectively, then the α_i and β_i-membership and non-membership degree of the vertex in the $\mathscr{K} - \mathscr{LMP}\ \mathscr{G}_1[\mathscr{G}_2]^{max}$ of $\mathscr{IFG}_{\mathscr{J}}$, $\mathscr{G}_1$ and $\mathscr{G}_2$ is given by

$$\alpha_i - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_i,u_j) = |\mathscr{V}_2|\left[\alpha_i' - \mathfrak{d}_{\mathscr{G}_1}(v_i)\right] + \mathscr{G}_2(u_j)C_1,$$

$$\beta_i - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_i,u_j) = |\mathscr{V}_2|\left[\beta_i' - \mathfrak{d}_{\mathscr{G}_1}(v_i)\right] + \mathscr{G}_2(u_j)C_2.$$

Remark 1 Given two normal $\mathscr{IFG}_{\mathscr{J}}$ $\mathscr{G}_1$ and $\mathscr{G}_2$, their $\mathscr{K} - \mathscr{LMP}\ \mathscr{G}_1[\mathscr{G}_2]^{max}$. The $\mathscr{IFG}_{\mathscr{J}}$ is not regular.

Example 5 Consider two $\mathscr{IFG}_{\mathscr{J}}$, $\mathscr{G}_1 = \langle \mu_\alpha,\eta_\alpha,\rho_1',\rho_2'\rangle$ and $\mathscr{G}_2 = \langle \mu_\beta,\eta_\beta,\rho_1'',\rho_2'',\rho_3'',\rho_4''\rangle$ as shown in Figure 9.

It is clear that from the Figure 9 that each vertex of $\mathscr{G}_1 = \langle \mu_\alpha,\eta_\alpha,\rho_1',\rho_2',\rho_3'\rangle$ has one β_i' edge with the same degree of member and non-membership respectively, that is (0.7, 0.2). Hence, $\mathscr{G}_1$ is a (0.7, 0.2) μ_α' and μ_β' is regular $\mathscr{IFG}_{\mathscr{J}}$. Moreover, each vertex of $\mathscr{G}_2 = \langle \mu_\beta,\eta_\beta,\rho_1'',\rho_2'',\rho_3'',\rho_4''\rangle$ has one β_i'' edge with the same degree of member and non-membership respectively, that is (0.6, 0.3). Hence, $\mathscr{G}_2$ is a (0.6, 0.3) μ_α' and μ_β' is regular $\mathscr{IFG}_{\mathscr{J}}$.

Figure 9: IFGS

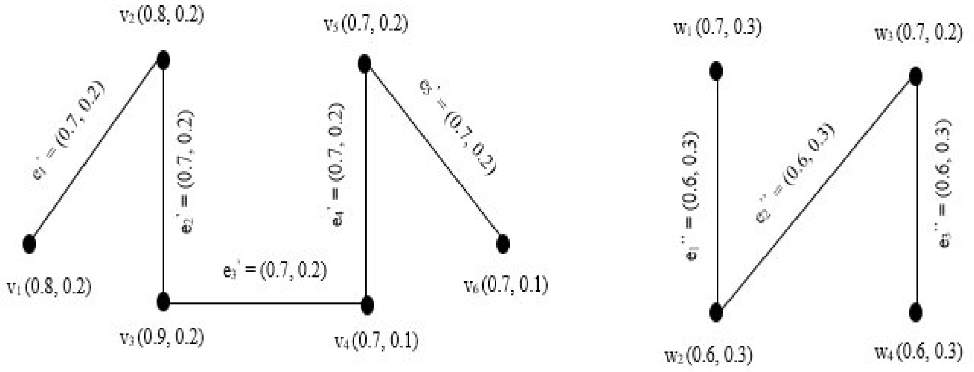
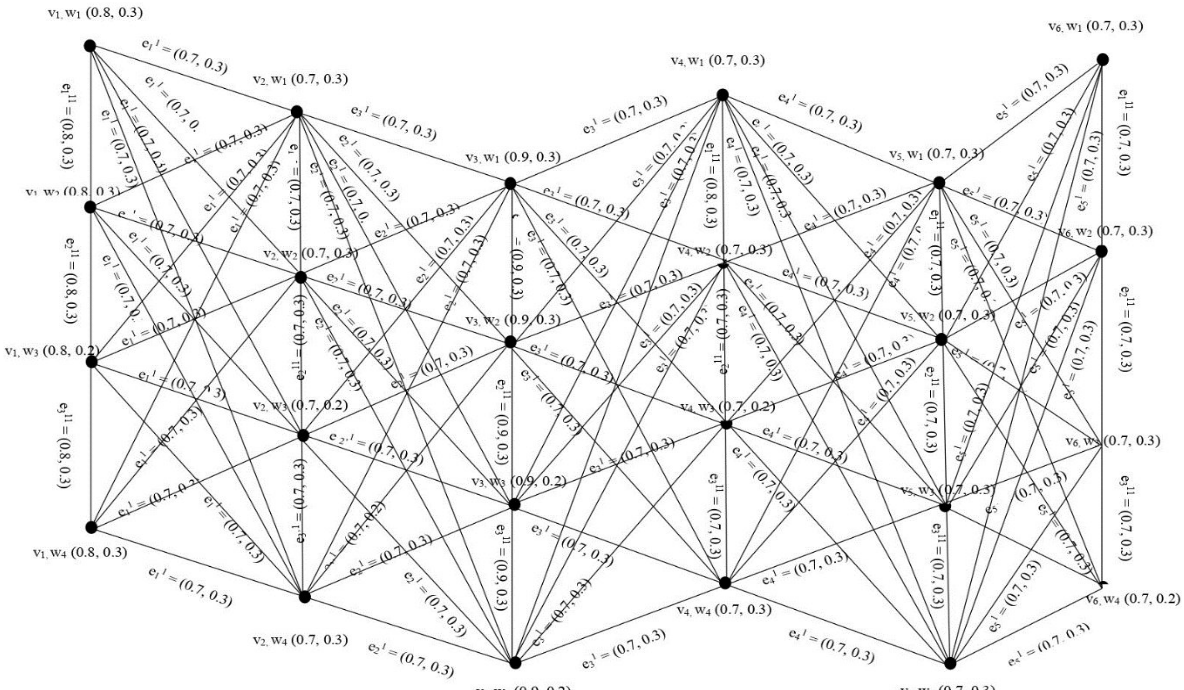


Figure 10: $\mathscr{K} - \mathscr{LMP}$ of IFGS



The $\mathscr{K}-\mathscr{LMP}$ $\mathscr{G}_1=\langle\mu_\alpha,\eta_\alpha,\rho'_1,\rho'_2,\rho'_3,\rho'_4,\rho'_5\rangle$ is shown in Figure 10. In Figure 9, all edges of $(\mu_\beta,\rho'_1,\rho'_2,\rho'_3,\rho'_4,\rho'_5)$ the membership and non-membership values (0.7,0.3). It is clear from Figure 10 that each vertex in the $\mathscr{K}-\mathscr{LMP}\mathscr{G}_1[\mathscr{G}_2]^{max}$ does not have any type of edge with the same membership and non-membership value. For example, The vertex (v_1w_1) has five ρ_1 edges, four ρ_1 edges with a (0.7,0.3) and one ρ_1 edge with a membership and non-membership value (0.8,0.3), while the vertex (v_3w_1) has seven ρ_1 edges, one ρ_1 edges with a (0.7,0.3) and one ρ_1 edge with a membership and non-membership value (0.9,0.3). Hence $\mathscr{G}_1[\mathscr{G}_2]^{max}$ is not a regular $\mathscr{IFG}_\mathscr{J}$.

Definition 9 The total degree of the $\mathscr{K}-\mathscr{LMP}$ $\mathscr{G}_1[\mathscr{G}_2]^{max}=\langle\mu_\alpha,\eta_\alpha,\rho_1,\rho_2,......\rho_n\rangle$ of the $\mathscr{IFG}_\mathscr{J}$, $\mathscr{G}_1=\langle\mu_\alpha,\eta_\alpha,\rho'_1,\rho'_2,.........\rho'_n\rangle$ and $\mathscr{G}_2=\langle\mu_\beta,\eta_\beta,\rho''_1,\rho''_2,.....\rho''_n\rangle$ for membership and non membership function μ and η is defined as

$$\tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_i,u_j)=\Sigma_{v_iv_k\in\mathscr{E}'_i u_l\in V_2}\alpha'_i(v_iv_k)+\Sigma_{u_ju_l\in\mathscr{E}''_j,v_i=v_k} min\left\{\alpha''_j(u_ju_l),\alpha_l(v_i)\right\}+\mu(u_i,v_j)$$

$$\tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_i,v_j)=\Sigma_{v_iv_k\in\mathscr{E}'_i u_l\in V_2}\beta'_i(v_iv_k)+\Sigma_{u_ju_l\in\mathscr{E}''_j,v_i=v_k} max\left\{\beta''_j(u_ju_l),\beta_l(v_i)\right\}+\eta(u_i,v_j)$$

α_i and β_i-the total degree of the $\mathscr{K}-\mathscr{LMP}\mathscr{G}_1[\mathscr{G}_2]^{max}$ is defined as

$$\alpha_i-\tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_i,u_j)=\Sigma_{v_iv_k\in\mathscr{E}'_i u_l\in V_2}\alpha'_i(v_iv_k)+\Sigma_{u_ju_l\in\mathscr{E}''_j,v_i=v_k} min\left\{\alpha''_j(u_ju_l),\alpha_l(v_i)\right\}+\frac{\mu(u_i,v_j)}{n}$$

$$\beta_i-\tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_i,v_j)=\Sigma_{v_iv_k\in\mathscr{E}'_i u_l\in V_2}\beta'_i(v_iv_k)+\Sigma_{u_ju_l\in\mathscr{E}''_j,v_i=v_k} max\left\{\beta''_j(u_ju_l),\beta_l(v_i)\right\}+\frac{\eta(u_i,v_j)}{n}$$

Example 6 Consider two $\mathscr{IFG}_\mathscr{J}$ $\mathscr{G}_1=\langle\mu_\alpha,\eta_\alpha,\rho'_1,\rho'_2\rangle$ and $\mathscr{G}_2=\langle\mu_\beta,\eta_\beta,\rho''_1,\rho''_2\rangle$ as show in Figure 11. The $\mathscr{K}-\mathscr{LMP}\mathscr{G}_1[\mathscr{G}_2]^{max}$ is shown in Figure 12. The total degree of vertices in the $\mathscr{K}-\mathscr{LMP}$ are computed as

Figure 11: IFGS

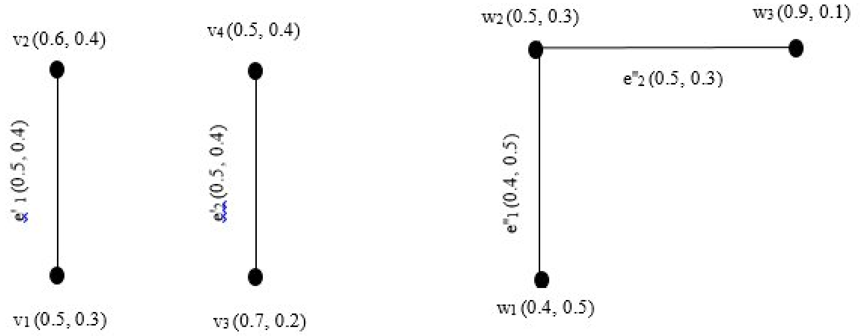
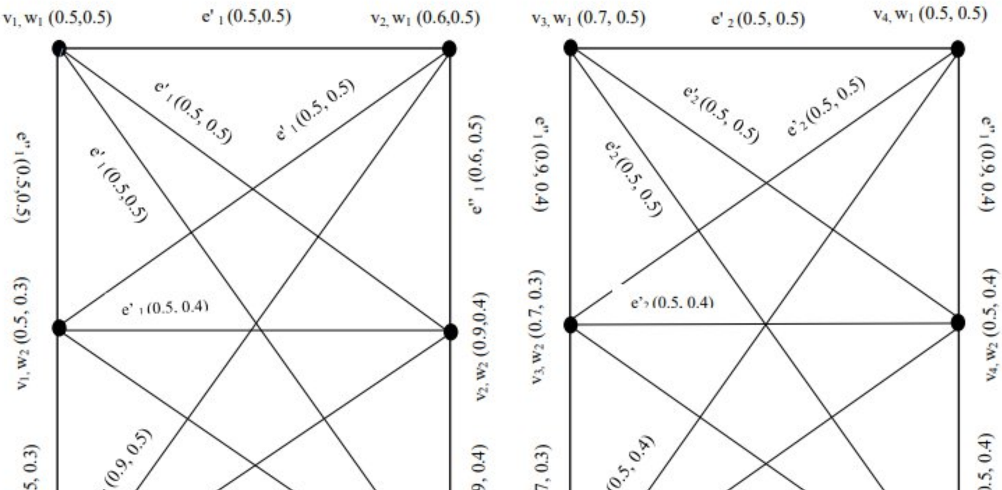


Figure 12: $\mathscr{K}-\mathscr{LMP}$ of IFGS



$\tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]}^{max}(v_i,u_j)$	Membership value	Non-Membership value
$\tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]}^{max}(v_1,u_1)$	2.4	2.5
$\tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]}^{max}(v_1,u_2)$	2.9	2.1
$\tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]}^{max}(v_1,u_3)$	3.5	1.8
$\tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]}^{max}(v_2,u_1)$	2.6	2.0
$\tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]}^{max}(v_2,u_2)$	3.1	2.7
$\tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]}^{max}(v_2,u_3)$	3.5	2.0
$\tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]}^{max}(v_3,u_1)$	2.8	2.5
$\tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]}^{max}(v_3,u_2)$	3.3	2.4
$\tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]}^{max}(v_3,u_3)$	3.7	1.7
$\tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]}^{max}(v_4,u_1)$	2.4	2.4
$\tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]}^{max}(v_4,u_2)$	02.9	2.6
$\tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]}^{max}(v_4,u_3)$	3.5	2.4

Table 6: Total degree of a vertex in the $\mathscr{K}-\mathscr{LMP}$

$$\tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]}^{max}(v_i,u_j) = \Sigma_{v_iv_k\in\mathscr{E}_i' u_l\in V_2}\alpha_i'(v_iv_k) + \Sigma_{u_ju_l\in\mathscr{E}_j'', v_i=v_k} min\left\{\alpha_j''(u_ju_l),\alpha_l(v_i)\right\} + \mu(u_i,v_j)$$

$$\tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]}^{max}(v_i,v_j) = \Sigma_{v_iv_k\in\mathscr{E}_i' u_l\in V_2}\beta_i'(v_iv_k) + \Sigma_{u_ju_l\in\mathscr{E}_j'', v_i=v_k} max\left\{\beta_j''(u_ju_l),\beta_l(v_i)\right\} + \eta(u_i,v_j)$$

Table 7: α_i and β_i be the degree of a vertex in the $\mathscr{K}-\mathscr{LMP}$

Vertices	$\mu_\alpha-\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]}^{max}(v_1,u_1)$	$\beta_i-\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]}^{max}(v_i,v_j)$
$\mu_\alpha-\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]}^{max}(v_1,u_1)$	3.6	2.25
$\mu_\alpha-\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]}^{max}(v_1,u_2)$	2.15	1.65
$\eta_\alpha-\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]}^{max}(v_1,u_2)$	0.75	0.45
$\mu_\alpha-\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]}^{max}(v_1,u_3)$	2.55	1.35
$\eta_\alpha-\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]}^{max}(v_1,u_3)$	0.94	0.45
$\mu_\alpha-\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]}^{max}(v_2,u_1)$	2.3	1.8
$\eta_\alpha-\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]}^{max}(v_2,u_1)$	0.3	0.2
$\mu_\alpha-\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]}^{max}(v_2,u_2)$	2.3	2.1
$\eta_\alpha-\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]}^{max}(v_2,u_2)$	0.8	0.6
$\mu_\alpha-\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]}^{max}(v_2,u_3)$	2.55	1.4
$\eta_\alpha-\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]}^{max}(v_2,u_3)$	0.95	0.6

$\mu_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_3,u_1)$	0.75	0.75
$\eta_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_3,u_1)$	2.05	1.75
$\mu_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_3,u_2)$	0.75	0.65
$\eta_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_3,u_2)$	2.55	1.75
$\mu_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_3,u_3)$	0.45	0.1
$\eta_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_3,u_3)$	2.65	1.6
$\mu_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_4,u_1)$	0.65	0.75
$\eta_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_4,u_1)$	1.75	1.65
$\mu_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_4,u_2)$	0.65	0.75
$\eta_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_4,u_2)$	2.25	1.9
$\mu_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_4,u_3)$	0.45	0.2
$\eta_\alpha - \mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_4,u_3)$	3.05	1.9

Example 7 It is clear from example 6 that

$$\left[\mu_\alpha - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_1,u_1)\right] + \left[\eta_\alpha - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_1,u_1)\right] = 2.15 + 0.25 = 2.4$$

$$\left[\mu_\beta - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_1,u_1)\right] + \left[\eta_\beta - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_1,u_1)\right] = 2.5$$

$$\left[\mu_\alpha - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_1,u_2)\right] + \left[\eta_\alpha - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_1,u_2)\right] = 2.15 + 0.75 = 2.9$$

$$\left[\mu_\beta - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_1,u_2)\right] + \left[\eta_\beta - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_1,u_2)\right] = 1.65 + 0.45 = 2.1$$

$$\left[\mu_\alpha - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_1,u_3)\right] + \left[\eta_\alpha - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_1,u_3)\right] = 2.55 + 0.95 = 3.5$$

$$\left[\mu_\beta - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_1,u_3)\right] + \left[\eta_\beta - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_1,u_3)\right] = 1.35 + 0.45 = 1.8$$

$$\left[\mu_\alpha - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_2,u_1)\right] + \left[\eta_\alpha - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_2,u_1)\right] = 2.3 + 0.3 = 2.6$$

$$\left[\mu_\beta - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_2,u_1)\right] + \left[\eta_\beta - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_2,u_1)\right] = 1.8 + 0.2 = 2$$

$$\left[\mu_\alpha - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_2,u_2)\right] + \left[\eta_\alpha - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_2,u_2)\right] = 2.3 + 0.8 = 3.1$$

$$\left[\mu_\beta - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_2,u_2)\right] + \left[\eta_\beta - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_2,u_2)\right] = 2.1 + 0.6 = 2.7$$

$$\left[\mu_\alpha - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_2,u_3)\right] + \left[\eta_\alpha - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_2,u_3)\right] = 2.5 + 0.95 = 3.5$$

$$\left[\mu_\beta - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_2,u_3)\right] + \left[\eta_\beta - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_2,u_3)\right] = 1.4 + 0.6 = 2$$

$$\left[\mu_\alpha - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_3,u_1)\right] + \left[\eta_\alpha - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_3,u_1)\right] = 2.05 + 0.75 = 2.8$$

$$\left[\mu_\beta - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_3,u_1)\right] + \left[\eta_\beta - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_3,u_1)\right] = 1.75 + 0.75 = 2.5$$

$$\left[\mu_\alpha - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_3,u_2)\right] + \left[\eta_\alpha - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_3,u_2)\right] = 2.55 + 0.75 = 3.3$$

$$\left[\mu_\beta - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_3,u_2)\right] + \left[\eta_\beta - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_3,u_2)\right] = 1.75 + 0.65 = 2.4$$

$$\left[\mu_\alpha - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_3,u_3)\right] + \left[\eta_\alpha - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_3,u_3)\right] = 2.65 + 0.45 = 3.3$$

$$\left[\mu_\beta - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_3,u_3)\right] + \left[\eta_\beta - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_3,u_3)\right] = 1.6 + 0.1 = 1.7$$

$$\left[\mu_\alpha - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_4,u_1)\right] + \left[\eta_\alpha - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_4,u_1)\right] = 1.75 + 0.65 = 2.4$$

$$\left[\mu_\beta - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_4,u_1)\right] + \left[\eta_\beta - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_4,u_1)\right] = 1.65 + 0.75 = 2.4$$

$$\left[\mu_\alpha - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_4,u_2)\right] + \left[\eta_\alpha - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_4,u_2)\right] = 2.25 + 0.65 = 2.9$$

$$\left[\mu_\beta - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_4,u_2)\right] + \left[\eta_\beta - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_4,u_2)\right] = 1.9 + \ + 0.7 = 2.6$$

$$\left[\mu_\alpha - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_4,u_3)\right] + \left[\eta_\alpha - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_4,u_3)\right] = 3.05 + 0.45 = 2.9$$

$$\left[\mu_\beta - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_4,u_3)\right] + \left[\eta_\beta - \tau\mathfrak{d}_{\mathscr{G}_1[\mathscr{G}_2]^{max}}(v_4,u_3)\right] = 1.9 + \ + 0.2 = 2.1$$

This example shows that the total degree of each vertex in $\mathscr{G}_1[\mathscr{G}_2]^{max}$ is equal to the sum of its α_i and β_i- total degree.

4. Application: Detection of Fraud in Online Transactions Using IFG

As the digital economy expands, the rise in the scope and number of online fraud is also increasing. Fraudulent activities such as identity theft, unauthorized access, and financial fraud have turned into notable matters of the safety and legitimacy of online transactions. These activities threaten not only individual users but also the stability of e-commerce platforms and financial institutions worldwide, potentially leading to significant economic losses and damage to consumer trust. Through digital commerce of worldwide nature, where there are numerous e-commerce platforms and payment getaways at the same time, the detection of fraudulent activities plays a principal role. Online frauds, including credit card scams, phishing, fraudulent chargebacks, and unauthorized access, have become increasingly sophisticated and need the most updated instruments that can detect and prevent early.

A real-life scenario of applying the IFG model could be used if an e-commerce platform decided to identify fraud transactions in the marketplace. It gathers and stores several details, including the activity details which include login frequency, browsing history, transaction amount, shipping address and payment methods. Standard fraud prevention methods can be rigid set procedural models, for example: transactions containing large values, or those originating from certain IP addresses. But these systems may not be able to identify other types of frauds which are more sophisticated such as account take overs as well as fake accounts that are similar to genuine accounts. With the help of IFG model the platform can build the dynamic fuzzy graph and each of the users and transactions can be described as the fuzzy nodes and edges. The K-Lexicographic-Max-Product technique is used in order to discover the suspicious connections between nodes and, thus, tentative fraudulent activities. The model is also flexible when new behaviors that were not in the initial flagged set are discovered, thus reducing on false negative in which genuine transactions are excluded.

IFG structures effectively model and analyze online fraud patterns across different platforms. An IFG models the relationships between various platforms and frequently used types of fraud between them, also accounting for a certain degree of hesitation or uncertainty in the detection of fraudulent activities. Let's assume a set V consisting of eight major e-commerce platforms: V={Amazon, eBay, Alibaba, Etsy, Shopify, PayPal, Stripe, Square}

Let α be an IFS on V with each having a membership value μ and a non-membership value η.

Table 8: $\mathscr{IFS}$ of Platforms

Platform	Membership	Non-Membership
Amazon	0.9	0.05
eBay	0.8	0.1
Alibaba	0.85	0.1
Etsy	0.7	0.15
Shopify	0.75	0.1
PayPal	0.85	0.05
Stripe	0.8	0.1
Square	0.75	0.1

Let the following relations represent different types of online fraud:

- R_1: Identity Theft
- R_2: Credit Card Fraud
- R_3: Phishing Attacks
- R_4: Unauthorized Access
- R_5: Fraudulent Chargebacks
- R_6: Money Laundering

The membership values μ signified the severity of a particular fraud type, while the non-membership values η indicate the degree to which that fraud type is less significant in the detection.

$\mathscr{IFS}$s for Fraud Types:

- $R_1 = \{(Amazon,PayPal),(Shopify,Stripe),(Amazon,Square),(eBay,PayPal)\}$
- $R_2 = \{(Amazon,eBay),(Stripe,Square)\}$

- $R_3 = \{(Shopify,Etsy),(Square,Stripe),(Alibaba,Etsy)\}$
- $R_4 = \{(PayPal,Stripe),(Amazon,Etsy),(Alibaba,Square)\}$
- $R_5 = \{(Amazon,Alibaba),(PayPal,Etsy),(Square,eBay)\}$
- $R_6 = \{(Alibaba,Stripe),(Amazon,Shopify),(Square,PayPal)\}$

The corresponding $\mathscr{IFS}$s

$$\rho_1 = \{((Amazon,PayPal),0.8,0.1),((Shopify,Stripe),0.7,0.2),((Amazon,Square),0.8,0.1),$$
$$((eBay,PayPal),0.6,0.3)\}$$

$$\rho_2 = \{((Amazon,eBay),0.9,0.05),((Stripe,Square),0.7,0.2)\}$$

$$\rho_3 = \{((Shopify,Etsy),0.7,0.2),((Square,Stripe),0.5,0.3),((Alibaba,Etsy),0.6,0.2)\}$$

$$\rho_4 = \{((PayPal,Stripe),0.85,0.1),((Amazon,Etsy),0.7,0.15),((Alibaba,Square),0.75,0.1)\}$$

$$\rho_5 = \{((Amazon,Alibaba),0.9,0.05),((PayPal,Etsy),0.7,0.2),((Square,eBay),0.6,0.3)\}$$

$$\rho_6 = \{((Alibaba,Stripe),0.8,0.1),((Amazon,Shopify),0.75,0.15),((Square,PayPal),0.7,0.2)\}$$

Table 9: Amazon and Others

Online Fraud	(Amazon, PayPal)	(Amazon, Square)	(Amazon, Alibaba)	(Amazon, eBay)
Identity Theft	(0.85, 0.05)	(0.75,0.1)	(0.25, 0.1)	(0.7, 0.15)
Credit Card Fraud	(0.75,0.10)	(0.5, 0.1)	(0.5, 0.2)	(0.6,0.3)
Phishing Attacks	(0.65,0.20)	(0.5,0.2)	(0.3,0.1)	(0.2,0.1)
Unauthorized Access	(0.7,0.2)	(0.3,0.1)	(0.6,0.2)	(0.5,0.5)
Fraudulent Chargebacks	(0.6,0.3)	(0.4,0.3)	(0.3,0.1)	(0.4,0.2)
Money Laundering	(0.5,0.4)	(0.6,0.2)	(0.5,0.3)	(0.6,0.1)

Table 10: eBay and Others

Online Fraud	(eBay, Shopify)	(eBay, Stripe)	(eBay, Etsy)	(eBay, Square)
Identity Theft	(0.6, 0.2)	(0.4,0.25)	(0.5, 0.35)	(0.6, 0.25)
Credit Card Fraud	(0.4,0.25)	(0.5, 0.3)	(0.6, 0.2)	(0.8,0.1)
Phishing Attacks	(0.6,0.2)	(0.8,0.1)	(0.7,0.25)	(0.65,0.05)
Unauthorized Access	(0.5,0.4)	(0.6,0.2)	(0.4,0.15)	(0.75,0.05)
Fraudulent Chargebacks	(0.7,0.25)	(0.55,0.15)	(0.45,0.2)	(0.9,0.05)
Money Laundering	(0.8,0.25)	(0.4,0.2)	(0.6,0.2)	(0.7,0.3)

Table 11: Alibaba and Others

Online Fraud	(Alibaba, Stripe)	(Alibaba, Square)	(Alibaba, Etsy)
Identity Theft	(0.5, 0.4)	(0.6,0.2)	(0.5, 0.25)
Credit Card Fraud	(0.7,0.15)	(0.8, 0.05)	(0.6, 0.35)
Phishing Attacks	(0.6,0.20)	(0.7,0.2)	(0.7,0.25)
Unauthorized Access	(0.4,0.1)	(0.3,0.2)	(0.9,0.1)
Fraudulent Chargebacks	(0.8,0.25)	(0.6,0.35)	(0.4,0.2)
Money Laundering	(0.75,0.15)	(0.3,0.2)	(0.3,0.15)

Table 12: Etsy and Others

Online Fraud	(Etsy, Shopify)	(Etsy, PayPal)	(Etsy, Square)
Identity Theft	(0.6, 0.25)	(0.7,0.3)	(0.5, 0.4)
Credit Card Fraud	(0.7,0.15)	(0.8, 0.2)	(0.6, 0.3)
Phishing Attacks	(0.8,0.25)	(0.6,0.3)	(0.45,0.25)
Unauthorized Access	(0.6,0.35)	(0.5,0.2)	(0.75,0.5)
Fraudulent Chargebacks	(0.4,0.2)	(0.4,0.1)	(0.6,0.2)
Money Laundering	(0.5,0.25)	(0.3,0.2)	(0.2,0.1)

Table 13: Shopify and Others

Online Fraud	(Shopify, Amazon)	(Shopify, Square)	(Shopify, PayPal)
Identity Theft	(0.5, 0.2)	(0.4,0.1)	(0.3, 0.2)
Credit Card Fraud	(0.4,0.2)	(0.6, 0.2)	(0.5, 0.3)
Phishing Attacks	(0.7,0.3)	(0.8,0.1)	(0.2,0.1)
Unauthorized Access	(0.8,0.1)	(0.6,0.2)	(0.4,0.2)
Fraudulent Chargebacks	(0.3,0.2)	(0.4,0.3)	(0.6,0.4)
Money Laundering	(0.5,0.4)	(0.7,0.2)	(0.9,0.1)

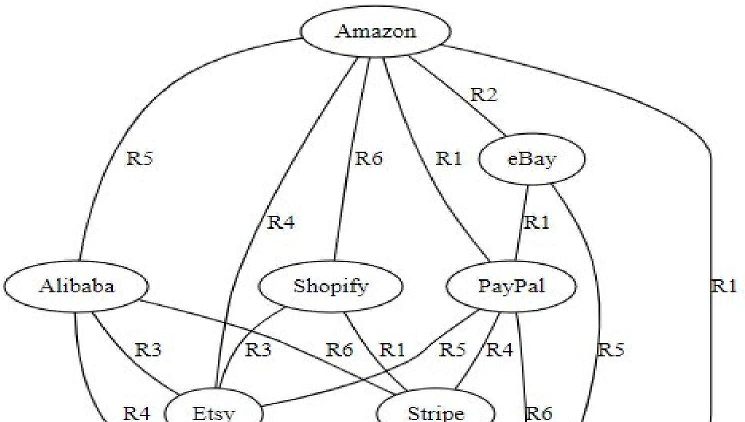
Table 14: PayPal and Others

Online Fraud	(PayPal, Alibaba)	(PayPal, eBay)	(PayPal, Amazon)
Identity Theft	(0.9, 0.1)	(0.8,0.2)	(0.4, 0.2)
Credit Card Fraud	(0.6,0.3)	(0.5, 0.4)	(0.6, 0.25)
Phishing Attacks	(0.5,0.4)	(0.6,0.3)	(0.5,0.35)
Unauthorized Access	(0.7,0.25)	(0.8,0.15)	(0.3,0.1)
Fraudulent Chargebacks	(0.35,0.1)	(0.7,0.1)	(0.6,0.2)
Money Laundering	(0.8,0.1)	(0.4,0.3)	(0.3,0.3)

Table 15: Some Others Platforms

Online Fraud	(Stripe, PayPal)	(Stripe, Shopify)	(Square, eBay)
Identity Theft	(0.3, 0.2)	(0.4,0.1)	(0.3, 0.1)
Credit Card Fraud	(0.7,0.4)	(0.6, 0.3)	(0.6, 0.4)
Phishing Attacks	(0.6,0.1)	(0.5,0.2)	(0.5,0.5)
Unauthorized Access	(0.5,0.4)	(0.6,0.2)	(0.9,0.1)
Fraudulent Chargebacks	(0.7,0.15)	(0.4,0.1)	(0.8,0.2)
Money Laundering	(0.6,0.4)	(0.9,0.1)	(0.2,0.1)

Figure 13: $\mathcal{IFG}$ for online Platform



4.1. Result Analysis:

The study of different frauds happening in the e-Commerce platforms through $\mathscr{IFG}$s provides a deeper understanding of the level of prescriptive and descriptive of various kinds of concrete frauds in different platforms. The membership values (μ) are interpreted as how much a certain type of fraud belongs to the specific platform, the non-membership values (η) in contrast denote how little a certain type of fraud belongs to this platform or can be detected on this platform. The analysis demonstrates the following key findings:

4.2 Severity of Fraud Types:

The membership values in the fraud group of Amazon, PayPal, and Stripe are quite high for serious frauds like Identity Theft and Unauthorized Access. For instance, Identity theft indexed on Amazon has a membership value of 0.85-a high level of severity. Credit Card Fraud and Money Laundering also exist in multiple platforms with large membership values are highly likely indicating that they are major issues in the e-commerce industry. For example, in Money Laundering involving PayPal and Alibaba got high values such as 0.8,0.1 hibited.

4.3 Platform Vulnerability:

The platforms which are most linked to different sorts of frauds according to the research are Amazon and PayPal, which have high membership values. This make them the main focus when it comes to issues of fraud control and prevention strategies. Though platforms like Square and Shopify have lower membership values for some types of fraud, both platforms are involved in Credit Card Fraud and Fraudulent Chargebacks, for example. This means that although these platforms are not the worst hit, they are equally vulnerable and should therefore be guarded. Alibaba retains a moderate to high membership value in all fraud categories including Money Laundering and Phishing Attacks and is evidently involved in different types of fraudulent practices. Fraud Detection and Prevention:

The $\mathscr{IFG}$ model is used to illustrate the intricacy of many platforms? fraud schemes. The hesitation values within these graphs similarly quantize the uncertainty in detecting fraud, which may be due to differing levels of security on a platform or different rates of fraudulent activity. With the help of intuitionistic fuzzy sets, one is able to differentiate platforms where the task of identifying fraud may prove difficult due to high uncertainty (hesitation) values. For instance, Phishing Attacks on Amazon we have a membership of 0.8 different types of claim readiness, with a score of 65 but non-membership of 0.20, meaning some level of uncertainty in detection.

As aforementioned, familiar frauds such as Phishing Attacks and Unauthorized Access have higher hesitation values on the platforms like Shopify or Etsy; it means that it is more challenging to identify these frauds on the aforementioned platforms, thus, need additional security measures.

4.4 Inter-Platform Fraud Analysis:

The tables indicate that fraudulent activities are inter-connected across the platforms where they were identified to operate. For example using the approach above, Credit Card Fraud which includes both Amazon and eBay has values of 0.9 and 0.05 meaning cross- platform fraud exists.

Skeen connections like PayPal and stripe, shopify and Square are examples of fraud, meaning that fraud is connected across multiple platforms, and it requires a joined up approach to address across multiple platforms across e-commerce.

5 Comparison Analysis:

The intuitionistic fuzzy graph model for fraud detection system in digital transactions based on K-lexicographic-max-product is a better method to tackle fraud compared to traditional method as it has the following benefits. In contrast to most rule-based and statistical models, which are not able to address the issues of ambiguity and changing patterns of fraud this model incorporates the Intuitionistic Fuzzy Logic in handling the ambiguity in the transaction data. It offers higher accuracy, increase in flexibility, and work scalability since it models interactions within the different variables and accommodates new fraud forms. Furthermore, it eliminates a significant number of false positives and also makes the results quite easy to interpret through use of fuzzy graphs. Compared with traditional static models that need to be manually reset/retrained for new fraud schemes and are not scalable for large volumes, high dimensions, the proposed intuitionistic fuzzy graph model self-evolves to new fraud schemes as the new data appears and is demonstrated to be computationally efficient. It makes it suitable for online fraud detection in real-time transactions, a better technique than the usual methods, in terms of flexibility and functionality.

6 Conclusion

The analysis of $\mathscr{IFG}$ brings out the magnitude and novelty of fraud in e-commerce platforms comprehensively. This emphasizes the need for sophisticated fraud detection tools capable of meeting the needed levels of automation while also dealing with the inherent levels of both epistemological and ontological risk in fraud type classifications. Sites such as Amazon, PayPal and Alibaba become instead primary targets and entities that need to have efficient, up-to-date anti-fraud measures. Since $\mathscr{IFG}$s allow hesitation and uncertainty in the structures, it can be said that the developed system provides e-commerce platforms and financial institutions with opportunities to prevent online fraud risks effectively.

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