Random Banach Stability Of Additive- Quadratic Mixed Type Functional Equation

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ABSTRACT
In this work, we address the generalized Ulam-Hyers stability using fixed point and direct methods of an additive-quadratic mixed type functional equation in Random Banach space.

Key words: Additive-quadratic functional equation, generalized Ulam-Hyers stability, Random Normed Space, direct method, fixed point method.

1. INTRODUCTION

One of the finest interesting question in the theory of functional equations is when is it true that a function, which approximately satisfies a functional equation must be close to an exact solution of the given functional equation?

If the problem accepts a unique solution, we say the equation is stable.

In 1940,Ulam [35] posed the famous Ulam stability problem. In 1941, D.H.Hyers[18] solved the well-known Ulam stability problem for additive mappings subject to the hyers condition on approximately additive mappings. He gave rise to the stability theory of functional equations. In 1950, T.Aoki [2] was the second author to treat this problem for additive mappings. In 1978, Rassias [27] provided a generalized version of Hyers for approximately linear mappings. In addition Rassias generalized the Hyers stability result by introducing two weaker conditions controlled by product of different powers of norms and a mixed product sum of powers of norms, respectively. A generalization of all the above stability results was obtained by P. Gavruta [15] in 1994 by replacing the unbounded Cauchy difference by a general control function $\varphi(x, y)$.

In 2008, a special case of Gavruta's theorem for the unbounded Cauchy difference was obtained by M.Arunkumaretal., [6] by considering the summation of both the sum and the product of two p-norms.
The functional equation

$$P(y_1 + y_2) = P(y_1) + P(y_2) \tag{1.1}$$

and

$$P(y_1 + y_2) + P(y_1 - y_2) = 2P(y_1) + 2P(y_2) \tag{1.2}$$

are known as Additive and Quadratic functional equations, respectively. Also, some different form of Quadratic functional equation is

$$P(2y_2 + y_1) + P(2y_2 - y_1) = P(y_2 + y_1) + P(y_2 - y_1) + 6P(y_2) \qquad (1.3)$$

was investigated by [8]. The general solution and generalized Hyers-Ulam stability for the Additive-Quadratic type functional equation is of the form

$$P(y_1 + ay_2) + aP(y_1 - y_2) = P(y_1 - ay_2) + aP(y_1 + y_2) \qquad (1.4)$$

was discussed by K.W.Jun and H.M.Kim [20]. Also, A.Najati and M.B.Moghimi [26] investigated the generalized Hyers-Ulam stability for the Additive-quadratic functional equation of the form

$$P(2y_1 + y_2) + P(2y_1 - y_2) = 2P(y_1 + y_2) + 2P(y_1 - y_2) + 2P(2y_1) - 4P(y_1) \quad (1.5)$$

Infact, the general solution and generalized Ulam-Hyers stability of a mixed type Additive-Quadratic functional equation

$$P(y_1 + y_2) + P(y_1 - y_2) = 2P(y_1) + P(y_2) + P(-y_2) \qquad (1.6)$$

was investigated by M.Arunkumar and J.M.Rassias [11]. Also, the general solution in vector space and generalized Ulam-Hyers stability of mixed type Arun-Additive Quadratic functional equation is of the form

$$\begin{aligned} P(2y_1 \pm y_2 \pm y_3) &= 2P(-y_1 \pm y_2 \pm y_3) - 2P(\pm y_2 \pm y_3) \\ &\quad + P(\pm y_2 \pm y_3) + 3P(y_1) - P(-y_2) \qquad (1.7) \end{aligned}$$

in Random normed space was discussed by M.Arunkumar et.al., [25].

In this work, we address the generalized Ulam-Hyers stability using fixed point and direct methods of an additive-quadratic mixed type functional equation

$$\begin{aligned} &P(4y_4 + 3y_3 + 2y_2 + y_1) + P(4y_4 - 3y_3 + 2y_2 + y_1) + P(4y_4 + 3y_3 - 2y_2 + y_1) \\ &\quad + P(4y_4 + 3y_3 + 2y_2 - y_1) + P(4y_4 - 3y_3 - 2y_2 + y_1) + P(4y_4 - 3y_3 + 2y_2 - y_1) \\ &\quad + P(4y_4 + 3y_3 - 2y_2 - y_1) + P(4y_4 - 3y_3 - 2y_2 - y_1) \\ &= 8\{P(y_4 + y_1) + P(y_4 - y_1)\} - 40\{P(y_3) + P(-y_3)\} \\ &\quad - 3\{P(y_4 + y_1) + P(-y_4 - y_1) + P(y_4 - y_1) + P(y_1 - y_4)\} \\ &\quad + 4\{P(y_4 + y_2) - P(-y_4 - y_2) + P(y_4 - y_2) - P(y_2 - y_4)\} \\ &\quad + 31\{P(y_4 + y_3) + P(-y_4 - y_3) + P(y_4 - y_3) + P(y_3 - y_4)\} \\ &\quad + P(y_2 + y_1) + P(-y_2 - y_1) + P(y_2 - y_1) + P(y_1 - y_2) \\ &\quad + 7\{P(y_3 + y_2) + P(-y_3 - y_2) + P(y_3 - y_2) + P(y_2 - y_3)\} \quad (1.8) \end{aligned}$$

in Random Banach space.

2. GENERAL SOLUTION

In this portion, the authors debate the general solution of functional equation (1.8). By considering K and L as real vector spaces.

Theorem 2.1. If an odd function $P : K \to L$ fulfilling with the functional equation (1.1) if and only if $P : K \to L$ fulfilling the functional equation(1.8) for all $y_4, y_3, y_2, y_1 \in K$.

Proof: Suppose $P : K \to L$ fulfilling with the functional equation (1.1).
Setting (y_1, y_2) by (0,0) in (1.1), we get $P(0) = 0$. Replacing $y_2 = y_1$ in (1.1), we have $P(2y_1) = 2P(y_1)$, for all $y_1 \in K$. Let $y_2 = 2y_1$ in (1.1), we acquire $P(3y_1) = 3P(y_1)$, for all $y_1 \in K$. In general for a positive integer N, such that $P(Ny_1) = NP(y_1)$, for all $y_1 \in K$. Replacing $y_1 = 4y_4 + 3y_3$ and $y_2 = 2y_2 + y_1$ in (1.1), we get

$$P(4y_4 + 3y_3 + 2y_2 + y_1) = 4P(y_4) + 3P(y_3) + 2P(y_2) + P(y_1) \qquad (2.1)$$

for all $y_4, y_3, y_2, y_1 \in K$. Substituting $y_1 = 4y_4 - 3y_3$ and $y_2 = 2y_2 + y_1$ in (1.1), we obtain

$$P(4y_4 - 3y_3 + 2y_2 + y_1) = 4P(y_4) - 3P(y_3) + 2P(y_2) + P(y_1) \qquad (2.2)$$

for all $y_4, y_3, y_2, y_1 \in K$. Replacing$y_1 = 4y_4 + 3y_3$ and $y_2 = -2y_2 + y_1$ in (1.1), we arrive

$$P(4y_4 + 3y_3 - 2y_2 + y_1) = 4P(y_4) + 3P(y_3) - 2P(y_2) + P(y_1) \qquad (2.3)$$

for all $y_4, y_3, y_2, y_1 \in K$. Plugging $y_1 = 4y_4 + 3y_3$ and $y_2 = 2y_2 - y_1$ in (1.1), we have

$$P(4y_4 + 3y_3 + 2y_2 - y_1) = 4P(y_4) + 3P(y_3) + 2P(y_2) - P(y_1) \qquad (2.4)$$

for all $y_4, y_3, y_2, y_1 \in K$. Substituting $y_1 = 4y_4 - 3y_3$ and $y_2 = -2y_2 + y_1$ in (1.1), we get

$$P(4y_4 - 3y_3 - 2y_2 + y_1) = 4P(y_4) - 3P(y_3) - 2P(y_2) + P(y_1) \qquad (2.5)$$

for all $y_4, y_3, y_2, y_1 \in K$. Replacing$y_1 = 4y_4 - 3y_3$ and $y_2 = 2y_2 - y_1$ in (1.1), we obtain

$$P(4y_4 - 3y_3 + 2y_2 - y_1) = 4P(y_4) - 3P(y_3) + 2P(y_2) - P(y_1) \quad (2.6)$$

for all $y_4, y_3, y_2, y_1 \in K$. Replacing$y_1 = 4y_4 + 3y_3$ and $y_2 = -2y_2 - y_1$ in (1.1), we have

$$P(4y_4 + 3y_3 - 2y_2 - y_1) = 4P(y_4) + 3P(y_3) - 2P(y_2) - P(y_1) \quad (2.7)$$

for all $y_4, y_3, y_2, y_1 \in K$. Replacing$y_1 = 4y_4 - 3y_3$ and $y_2 = -2y_2 - y_1$ in (1.1), we get

$$P(4y_4 - 3y_3 - 2y_2 - y_1) = 4P(y_4) - 3P(y_3) - 2P(y_2) - P(y_1) \quad (2.8)$$

for all $y_4, y_3, y_2, y_1 \in K$. Adding all the equations (2.1), (2.2), (2.3), (2.4), (2.5), (2.6), (2.7), (2.8) , we arrive

$$P(4y_4 + 3y_3 + 2y_2 + y_1) + P(4y_4 - 3y_3 + 2y_2 + y_1) + P(4y_4 + 3y_3 - 2y_2 + y_1) + P(4y_4 + 3y_3 + 2y_2 - y_1) + P(4y_4 - 3y_3 - 2y_2 + y_1) + P(4y_4 - 3y_3 + 2y_2 - y_1) + P(4y_4 + 3y_3 - 2y_2 - y_1) + P(4y_4 - 3y_3 - 2y_2 - y_1) = 32P(y_4) \quad (2.9)$$

for all $y_4, y_3, y_2, y_1 \in K$. with the help of odd function from the above equation transformed as

$$\begin{aligned} &P(4y_4 + 3y_3 + 2y_2 + y_1) + P(4y_4 - 3y_3 + 2y_2 + y_1) + P(4y_4 + 3y_3 - 2y_2 + y_1) \\ &+ P(4y_4 + 3y_3 + 2y_2 - y_1) + P(4y_4 - 3y_3 - 2y_2 + y_1) + P(4y_4 - 3y_3 + 2y_2 - y_1) \\ &+ P(4y_4 + 3y_3 - 2y_2 - y_1) + P(4y_4 - 3y_3 - 2y_2 - y_1) \\ &= 8\{P(y_4 + y_1) + P(y_4 - y_1)\} - 40\{P(y_3) + P(-y_3)\} \\ &\quad - 3\{P(y_4 + y_1) + P(-y_4 - y_1) + P(y_4 - y_1) + P(y_1 - y_4)\} \\ &+4\{P(y_4 + y_2) - P(-y_4 - y_2) + P(y_4 - y_2) - P(y_2 - y_4)\} \\ &+31\{P(y_4 + y_3) + P(-y_4 - y_3) + P(y_4 - y_3) + P(y_3 - y_4)\} \\ &+P(y_2 + y_1) + P(-y_2 - y_1) + P(y_2 - y_1) + P(y_1 - y_2) \\ &+7\{P(y_3 + y_2) + P(-y_3 - y_2) + P(y_3 - y_2) + P(y_2 - y_3)\} \quad (2.10) \end{aligned}$$

for all $y_4, y_3, y_2, y_1 \in K$. It follows from (2.10), we have demonstrated our result (1.8).

Conversely, let $P : K \to L$ fulfilling with the functional equation (1.8) with $P(0) = 0$. Setting $(y_4, y_3, y_2, y_1) = (y_4, y_3, 0,0)$ and apply oddness in (1.8), we have

$$P(4y_4 + 3y_3) + P(4y_4 - 3y_3) = 8\big(P(y_4)\big) \quad (2.11)$$

for all $y_4, y_3 \in K$. Putting $y_3 = 0$ in (2.11), we have

$$P(4y_4) = 4P(y_4)$$

for all $y_2, y_1 \in K$. Putting $y_4 = \frac{y_1}{4}$ and $y_3 = \frac{y_2}{3}$in (2.11), we acquire

$$P(y_1 + y_2) + P(y_1 - y_2) = 2P(y_1) \quad (2.12)$$

for all $y_2, y_1 \in K$. Interchanging y_1 and y_2 in above equation , we obtain

$$P(y_2 + y_1) + P(y_2 - y_1) = 2P(y_2) \quad (2.13)$$

It follows from (2.12) that

$$P(y_1 + y_2) - P(y_2 - y_1) = 2P(y_1) \quad (2.14)$$

Adding (2.13) and (2.14), we arrive our desired result.

Theorem 2.2. If an even function $P : K \to L$ fulfilling with the functional equation (1.2) if and only if $P : K \to L$ fulfilling the functional equation (1.8) for all $y_4, y_3, y_2, y_1 \in K$.

Proof: Suppose $P : K \to L$ fulfilling with the functional equation (1.2). Setting (y_1, y_2) by (0,0) in (1.2), we get $P(0) = 0$. Replacing $y_2 = y_1$ in (1.2), we have $P(2y_1) = 4P(y_1)$, for all $y_1 \in K$. Let $y_2 = 2y_1$ in (1.2), we acquire $P(3y_1) = 9P(y_1)$, for all $y_1 \in K$. In general for a positive integer N, such that $P(Ny_1) = N^2P(y_1)$, for all $y_1 \in K$. Replacing $y_1 = 4y_4 + 3y_3$ and $y_2 = 2y_2 + y_1$ in (1.2), we get

$$P\,(4y_4 + 3y_3 + 2y_2 + y_1) + P\,(4y_4 + 3y_3 - 2y_2 - y_1) = 2P(4y_4 + 3y_3) + 2P(2y_2 + y_1) \quad (2.15)$$

for all $y_4, y_3, y_2, y_1 \in K$. Substituting $y_1 = 4y_4 - 3y_3$ and $y_2 = 2y_2 + y_1$ in (1.2), we obtain

$$P\,(4y_4 - 3y_3 + 2y_2 + y_1) + P\,(4y_4 - 3y_3 - 2y_2 - y_1) = 2P(4y_4 - 3y_3) + 2P(2y_2 + y_1) \quad (2.16)$$

for all $y_4, y_3, y_2, y_1 \in K$. Replacing $y_1 = 4y_4 + 3y_3$ and $y_2 = -2y_2 + y_1$ in (1.2), we arrive

$$P\,(4y_4 + 3y_3 - 2y_2 + y_1) + P\,(4y_4 + 3y_3 + 2y_2 - y_1) = 2P(4y_4 + 3y_3) + 2P(-2y_2 + y_1) \quad (2.17)$$

for all $y_4, y_3, y_2, y_1 \in K$. Plugging $y_1 = 4y_4 + 3y_3$ and $y_2 = 2y_2 - y_1$ in (1.2), we have
$P\ (4y_4 - 3y_3 - 2y_2 + y_1) + P\ (4y_4 - 3y_3 + 2y_2 - y_1)$
$= 2P(4y_4 - 3y_3) + 2P(-2y_2 + y_1)$ (2.18)
for all $y_4, y_3, y_2, y_1 \in K$. Adding all the equations (2.15), (2.16), (2.17), (2.18), we arrive

$$P\ (4y_4 + 3y_3 + 2y_2 + y_1) + P\ (4y_4 - 3y_3 + 2y_2 + y_1) + P\ (4y_4 + 3y_3 - 2y_2 + y_1)$$
$$+P(4y_4 + 3y_3 + 2y_2 - y_1) + P\ (4y_4 - 3y_3 - 2y_2 + y_1) + P\ (4y_4 - 3y_3 + 2y_2 - y_1)$$
$$+P(4y_4 + 3y_3 - 2y_2 - y_1) + P(4y_4 - 3y_3 - 2y_2 - y_1)$$
$$= 128P(y_4) + 72P(y_3) + 32P(y_2) + 8(y_1) \quad (2.19)$$

for all $y_4, y_3, y_2, y_1 \in K$. Using eveness of function from the above equation transformed as

$$P\ (4y_4 + 3y_3 + 2y_2 + y_1) + P\ (4y_4 - 3y_3 + 2y_2 + y_1) + P\ (4y_4 + 3y_3 - 2y_2 + y_1)$$
$$+ P(4y_4 + 3y_3 + 2y_2 - y_1) + P\ (4y_4 - 3y_3 - 2y_2 + y_1) + P\ (4y_4 - 3y_3 + 2y_2 - y_1)$$
$$+P(4y_4 + 3y_3 - 2y_2 - y_1) + P(4y_4 - 3y_3 - 2y_2 - y_1)$$
$$= 62\{P(y_4 + y_3) + P(y_4 - y_3)\} + P(y_2 + y_1) + P\big(-(y_2 + y_1)\big) - 80P(y_3)$$
$$+14\{\ P(y_3 + y_2) + P(y_3 - y_2)\} - 3\{P(y_4 - y_1) + P(-(y_4 - y_1))\}$$
$$+8\{\ P(y_4 + y_1) + P(y_4 - y_1)\} - 3\{P(y_4 + y_1) + P(-(y_4 + y_1))\}$$
$$+31\{\ P(y_4 + y_3) + P(-(y_4 + y_3)) + P(y_4 - y_3) + P(-(y_4 - y_3)\}$$
$$+2\{P(y_2 + y_1) + P(y_2 - y_1)\} + 2\{P(y_1 + y_4) + P(y_1 - y_4)\}$$
$$+4\{\ P(y_4 + y_2) - P(-(y_4 + y_2)) + P(y_4 - y_2) - P(-(y_4 - y_2))\} \quad (2.20)$$

for all $y_4, y_3, y_2, y_1 \in K$. It follows from (2.20), we have demonstrated our result (1.8).

Conversely, let $P : K \to L$ fulfilling with the functional equation (1.8) with $P(0) = 0$. Setting $(y_4, y_3, y_2, y_1) = (0,0, y, 0)$ and apply evenness in (1.8), we have

$$P(2y) = 4P(y) \quad (2.21)$$

for all $y \in K$. Setting $(y_4, y_3, y_2, y_1) = (0, y, 0,0)$ and apply evenness in (1.8), we have

$$P(3y) = 9P(y) \quad (2.22)$$

for all $y \in K$. In general for a positive integer N, such that $P(Ny_1) = N^2P(y_1)$, for all $y_1 \in K$. Setting $(y_4, y_3, y_2, y_1) = (y_4, 0, y_2, 0)$ and apply evenness in (1.8), we have

$$P(4y_4 + 2y_2) + P(4y_4 - 2y_2) = 32P(y_4) + 8P(y_2) \quad (2.23)$$

for all $y_4, y_2 \in K$. Pasting $y_4 = \frac{y_1}{4}$ and $y_2 = \frac{y_2}{2}$in (2.23), we achieve that equation (1.2).

3. BASICS OF RN-SPACE

In the sequel, we adopt the usual terminology, notations and conventions of the theory of random normed spaces as in [23]. Hereafter, this paper, Δ^+ is the space ofdistribution functions, that is the space of all mappings $F: R \cup \{-\infty, +\infty\} \to [0,1]$ suchthat F is leftcontinuous and nondecreasing on R, $F(0) = 0$ and $F(+\infty) = 1$. $D^+$is asubset of Δ^+ consisting of all functions $F \in \Delta^+$ for which$l^-F(+\infty) = 1$, where $l^-f(x)$denotes the left limit of the function f at the point x,$l^-f(x) = \lim_{t\to x} f(t)$. The space$\Delta^+$is partially ordered by the usual point-wise ordering of functions, i.e.,$F \le G$if and only if $F(t) \le G(t)$for all $t \in R$. The maximal element for Δ^+ in this order is the d.f. given by:

$$\epsilon_0(t) = \begin{cases} 0, & if\ t \le 0, \\ 1, & if\ t > 0. \end{cases}$$

Definition 3.1. A mapping $T : [0,1] \times [0,1] \to [0,1]$ is called a continuous triangular norm (briefly, a continuous$t$ $-$norm) if T satisfies the following conditions:

(a) $T$is commutative and associative;
(b) $T$is continuous;
(c) $T(a, 1) = a$for all$a \in [0,1]$;
(d) $T(a, b) \le T(c, d)$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0,1]$.

Definition 3.2. A random normed space (briefly, RN-space) is a triple (X, μ, T), where X is a vector space, T is a continuous$t -$ norm and μ is a mapping from X into D^+satisfying the following conditions:

(RN1) $\mu_x(t) = \varepsilon(t)$ for all $t > 0$ if and only if $x = 0$;

(RN2) $\mu_{\alpha x}(t) = \mu_x(t/|\alpha|)$ for all $x \in X$, and $\alpha \in R$ with $\alpha \neq 0$;
(RN3) $\mu_{x+y}(t+s) \geq T\left(\mu_x(t), \mu_y(s)\right)$ for all $x, y \in X$ and $t, s \geq 0$.

Example 3.3. Every normed spaces $(X, \|\cdot\|)$ defines a random normed space (X, μ, T_M), where

$$\mu_x(t) = \frac{t}{t + \|x\|}$$

and T_M is the minimum $t-$ norm. This space is called the induced random normed space.

Definition 3.4. Let (X, μ, T) be a RN-space. A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for any $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x}(\varepsilon) > 1 - \lambda$ for all $n \geq N$.

Definition 3.5. A sequence $\{x_n\}$ in X is called a Cauchy sequence if, for any $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x_m}(\varepsilon) > 1 - \lambda$ for all $n \geq m \geq N$.

Definition 3.6. A RN-space (X, μ, T) is said to be complete if every Cauchy sequence in X is convergent to a point in X.

Theorem 3.7. If (X, μ, T) is a RN-space and $\{x_n\}$ is a sequence in X such that $x_n \to x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$ almost everywhere.

4. STABILITY RESULTS IN RN-SPACE

In this section the generalized Ulam-Hyers stability of a Additive-Quadratic mixed type functional equation (1.8) in RN-space is provided.

Hereafter, Let us consider K to be a linear space and (L, Ξ, T) to be a complete RN-space. Define a mapping $P : K \to L$ by

$$\begin{aligned}
GP(y_4, y_3, y_2, y_1) = \\
&P\,(4y_4 + 3y_3 + 2y_2 + y_1) + P\,(4y_4 - 3y_3 + 2y_2 + y_1) + P\,(4y_4 + 3y_3 - 2y_2 + y_1) \\
&+ P(4y_4 + 3y_3 + 2y_2 - y_1) + P\,(4y_4 - 3y_3 - 2y_2 + y_1) + P\,(4y_4 - 3y_3 + 2y_2 - y_1) \\
&+P(4y_4 + 3y_3 - 2y_2 - y_1) + P(4y_4 - 3y_3 - 2y_2 - y_1) - 8\{P(y_4 + y_1) + P(y_4 - y_1)\} \\
&+3\{P(y_4 + y_1) + P(-y_4 - y_1)\} + 3\{P(y_4 - y_1) + P(y_1 - y_4)\} \\
&-4\{P(y_4 + y_2) - P(-y_4 - y_2) + P(y_4 - y_2) - P(y_2 - y_4)\} \\
&-31\{P(y_4 + y_3) + P(-y_4 - y_3) + P(y_4 - y_3) + P(y_3 - y_4)\} \\
&-7\{P(y_3 + y_2) + P(-y_3 - y_2) + P(y_3 - y_2) + P(y_2 - y_3)\} - P(y_2 + y_1) \\
&-P(-y_2 - y_1) - P(y_2 - y_1) - P(y_1 - y_2) + 40\{P(y_3) + P(-y_3)\}
\end{aligned}$$

for all $y_4, y_3, y_2, y_1 \in K$.

4.1. Direct Method :

Theorem 4.1. Assume that $u \in \{\pm 1\}$. Let K be a linear space, (L, Ξ, T) be a complete RN-Space and $P_a : K \to L$ be an odd mapping for which there exists a function $\Lambda : K^4 \to D^+$ with the inequality

$$\Xi_{GP_a(y_4,y_3,y_2,y_1)}(c) \geq \Lambda_{y_4,y_3,y_2,y_1}(c) \tag{4.1}$$

for all $y_4, y_3, y_2, y_1 \in K$ and all $c > 0$. If

$$\lim_{m\to\infty} T_{l=0}^{\infty} \Lambda_{10^{ul}y,10^{ul}y,10^{ul}y,10^{ul}y}\left(10^{u(l+1)}c\right) = 1 = \lim_{m\to\infty} T_{l=0}^{\infty} \Lambda_{10^{ul}y_4,10^{ul}y_3,10^{ul}y_2,10^{ul}y_1}\left(10^{ul}c\right) \tag{4.2}$$

for all $y_4, y_3, y_2, y_1 \in K$ and all $c > 0$. Then there exists a unique additive mapping $A : K \to L$ such that

$$\Xi_{A(y)-P_a(y)}(5c) \geq T_{l=0}^{\infty} \Lambda^{a}_{10^{ul}y}\left(10^{u(l+1)}c\right) \tag{4.3}$$

for all y $\in K$ and all $c > 0$.

Proof: Assume $u = 1$. Substituting (y_4, y_3, y_2, y_1) with (y, y, y, y) in (4.1) and apply oddness, we get

$$\Xi_{P_a(10y)+P_a(8y)+P_a(6y)+2P_a(4y)-16P_a(2y)}(c) \geq \Lambda_{y,y,y,y}(c) \qquad (4.4)$$

for all $y \in K$ and all $c > 0$. Again Substituting (y_4, y_3, y_2, y_1) with $(y, y, 0, -y)$ in (4.1) and apply oddness, we obtain

$$\Xi_{P_a(8y)+P_a(6y)-3P_a(2y)-8P_a(4y)-16P_a(2y)}\left(\frac{c}{2}\right) \geq \Lambda_{y,y,0,-y}(c) \qquad (4.5)$$

for all $y \in K$ and all $c > 0$. Again Substituting (y_4, y_3, y_2, y_1) with $(y_4, y_3, 0,0)$ in (4.1) and apply oddness, we get

$$\Xi_{P_a(4y_4+3y_3)+P_a(4y_4-3y_3)-8P_a(y_4)}\left(\frac{c}{4}\right) \geq \Lambda_{y_4,y_3,0,0}(c) \qquad (4.6)$$

for all $y \in K$ and all $c > 0$. Put $y_4 = \frac{y}{4}$ and $y_3 = \frac{y}{3}$ in above inequality and using $(RN2)$, we acquire

$$\Xi_{13P_a(2y)-26P_a(y)}\left(\frac{13c}{4}\right) \geq \Lambda_{\frac{y}{4},\frac{y}{3},0,0}(c) \qquad (4.7)$$

for all $y \in K$ and all $c > 0$. Again substituting (y_4, y_3, y_2, y_1) with $(y, 0,0,0)$ in (4.1) and apply oddness, we get

$$\Xi_{2P_a(4y)-8P_a(y)}\left(\frac{c}{4}\right) \geq \Lambda_{y,0,0,0}(c) \qquad (4.8)$$

for all $y \in K$ and all $c > 0$. It follows from (4.4) and (4.5), we have

$$\Xi_{P_a(10y)+2P_a(4y)-13P_a(2y)+8P_a(y)}\left(\frac{3c}{2}\right) \geq T\left(\Lambda_{y,y,y,y}(c), \Lambda_{y,y,0,-y}(c)\right) \qquad (4.9)$$

for all $y \in K$ and all $c > 0$. It follows from (4.8) and (4.9), we obtain

$$\Xi_{P_a(10y)-13P_a(2y)+16P_a(y)}\left(\frac{7c}{4}\right) \geq T\left(T\left(\Lambda_{y,y,y,y}(c), \Lambda_{y,y,0,-y}(c)\right), \Lambda_{y,0,0,0}(c)\right) \qquad (4.10)$$

for all $y \in K$ and all $c > 0$. It follows from (4.7) and (4.10), we arrive

$$\Xi_{P_a(10y)-10P_a(y)}(5c) \geq T\left(T\left(T\left(\Lambda_{y,y,y,y}(c), \Lambda_{y,y,0,-y}(c)\right), \Lambda_{y,0,0,0}(c)\right), \Lambda_{\frac{y}{4},\frac{y}{3},0,0}(c)\right) \qquad (4.11)$$

for all $y \in K$ and all $c > 0$. It follows from (4.11) that

$$\Xi_{P_a(10y)-10P_a(y)}(5c) \geq \Lambda_y^a(c) \qquad (4.12)$$

where $\Lambda_y^a(c) = T\left(T\left(T\left(\Lambda_{y,y,y,y}(c), \Lambda_{y,y,0,-y}(c)\right), \Lambda_{y,0,0,0}(c)\right), \Lambda_{\frac{y}{4},\frac{y}{3},0,0}(c)\right)$

for all $y \in K$ and all $c > 0$. It follows from (4.12) and $(RN2)$, we obtain

$$\Xi_{\frac{P_a(10y)}{10}-P_a(y)}(5c) \geq \Lambda_y^a(10c) \qquad (4.13)$$

for all $y \in K$ and all $c > 0$. Replacing y by $10^n y$ in above inequality, we arrive

$$\Xi_{\frac{P_a(10^{n+1}y)}{10^{n+1}}-\frac{P_a(10^n y)}{10^n}}(5c) \geq \Lambda_{10^n y}^a(10^{n+1}c) \qquad (4.14)$$

for all $y \in K$ and all $c > 0$. It is easy to see that

$$\frac{P_a(10^n y)}{10^n} - P_a(y) = \sum_{l=0}^{n-1}\left(\frac{P_a(10^{l+1}y)}{10^{l+1}} - \frac{P_a(10^l y)}{10^l}\right) \qquad (4.15)$$

for all $y \in K$ and all $c > 0$. From the equations (4.14) and (4.15), we have

$$\begin{aligned}\Xi_{\frac{P_a(10^n y)}{10^n}-P_a(y)}(5c) &= \Xi_{\sum_{l=0}^{n-1}\left(\frac{P_a(10^{l+1}y)}{10^{l+1}}-\frac{P(10^l y)}{10^l}\right)}(c)\\ &\geq T_{l=0}^{n-1}\Xi_{\sum_{l=0}^{n-1}\left(\frac{P_a(10^{l+1}y)}{10^{l+1}}-\frac{P_a(10^l y)}{10^l}\right)}(10^{l+1}c)\\ &\geq T_{l=0}^{n-1}\Lambda_{10^l y}^a(10^{l+1}c) \qquad (4.16)\end{aligned}$$

for all $y \in K$ and all $c > 0$. In order to prove the convergence of the sequence $\left\{\frac{P_a(10^n y)}{10^n}\right\}$, first substituting y by $10^m y$ in (4.16), we achieve

$$\Xi_{\frac{P_a(10^{n+m}y)}{10^{n+m}}-\frac{P_a(10^m y)}{10^m}}(5c) \geq T_{l=0}^{n-1}\Lambda^a_{10^{l+m}y}\left(10^{l++m+1}c\right)$$
$$\geq T_{l=0}^{m+n-1}\Lambda^a_{10^l y}\left(10^{l+1}c\right)$$
$$\to 1 \text{ as } m \to \infty \qquad (4.17)$$

for all $y \in K$ and all $c > 0$. Thus $\left\{\frac{P_a(10^n y)}{10^n}\right\}$ is a Cauchy sequence. Since L is complete there exists a mapping $A : K \to L$, we define

$$\Xi_{A(y)}(c) = \lim_{l\to\infty} \Xi_{\frac{P_a(10^m y)}{10^m}}(c) \qquad (4.18)$$

for all $y \in K$ and all $c > 0$. Letting $m = 0$ and $n \to \infty$ in (4.16), we arrive (4.3) for all $y \in K$ and all $c > 0$. To prove that A satisfies (1.8), Replacing (y_4, y_3, y_2, y_1) by $(10^n y_4, 10^n y_3, 10^n y_2, 10^n y_1)$ and using $(RN2)$ in (4.1), we obtain

$$\Xi_{\frac{GP_a(10^n y_4, 10^n y_3, 10^n y_2, 10^n y_1)}{10^n}}(c) \geq \Lambda_{10^n y_4, 10^n y_3, 10^n y_2, 10^n y_1}(10^n c) \qquad (4.19)$$

for all $y_4, y_3, y_2, y_1 \in K$. Letting $n \to \infty$ in the above inequality and using the definition of $A(y)$, we see that A satisfies (1.8) for all $y_4, y_3, y_2, y_1 \in K$. Therefore the mapping A is Additive.

Finally, to prove the uniqueness of the additive function A, let us assume that there exists a additive function A' which satisfies (1.8). Since $A(10^n y) = 10^n A(y)$ and $A'(10^n y) = 10^n A'(y)$ for all $y \in K$ and all $c > 0$, it follows from (4.18) that

$$\Xi_{A(y)-A'(y)}(2c) = \Xi_{A(10^n y)-A'(10^n y)}(2.\,10^n c)$$
$$= \Xi_{A(10^n y)-P_a(10^n y)+P_a(10^n y)-A'(10^n y)}(2.\,10^n c)$$
$$\geq T\left[T_{l=0}^{\infty}\Lambda_{10^{l+n}y}\left(10^{l+n+1}c\right), T_{l=0}^{\infty}\Lambda_{10^{l+n}y}\left(10^{l+n+1}c\right)\right]$$
$$\to 1 \; as \; n \to \infty$$

for all $y \in K$ and all $c > 0$. Hence A is unique.

For $u = -1$, Plugging $y = \frac{y}{10}$ in (4.12), we have

$$\Xi_{P_a(y)-10P_a\left(\frac{y}{10}\right)}(5c) \geq \Lambda^a_{\frac{y}{10}}(c) \qquad (4.20)$$

for all $y \in K$ and all $c > 0$. Replacing y by $\frac{y}{10^n}$ in above inequality, we arrive

$$\Xi_{10^n P_a\left(\frac{y}{10^n}\right)-10^{n+1}P_a\left(\frac{y}{10^{n+1}}\right)}(5c) \geq \Lambda^a_{\frac{y}{10^{n+1}}}\left(\frac{c}{10^n}\right) \qquad (4.21)$$

for all $y \in K$ and all $c > 0$. It is easy to see that

$$P_a(y) - 10^n P_a\left(\frac{y}{10^n}\right) = \sum_{l=0}^{n-1}\left(10^l P_a\left(\frac{y}{10^l}\right) - 10^{l+1}P_a\left(\frac{y}{10^{l+1}}\right)\right) \qquad (4.22)$$

for all $y \in K$ and all $c > 0$. From the equations (4.21) and (4.22), we have

$$\Xi_{P_a(y)-10^n P_a\left(\frac{y}{10^n}\right)}(5c) = \Xi_{\sum_{l=0}^{n-1}\left(10^l P_a\left(\frac{y}{10^l}\right)-10^{l+1}P_a\left(\frac{y}{10^{l+1}}\right)\right)}(c)$$
$$\geq T_{l=0}^{n-1}\Xi_{\sum_{l=0}^{n-1}\left(\frac{P_a(10^{l+1}y)}{10^{l+1}}-\frac{P_a(10^l y)}{10^l}\right)}\left(\frac{c}{10^l}\right)$$
$$\geq T_{l=0}^{n-1}\Lambda^a_{\frac{y}{10^{l+1}}}\left(\frac{c}{10^l}\right) \qquad (4.23)$$

for all $y \in K$ and all $c > 0$. The rest of the proof is similar to that of $u = 1$. This completes the proof of the theorem.

The following Corollary is an immediate consequence of Theorem 4.1 concerning the stability of (1.8).

Corollary 4.2. Let β and s be nonnegative real numbers. Let an odd function $P_a : K \to L$ satisfies the inequality

$$\Xi_{GP_a(y_4,y_3,y_2,y_1)}(c) \geq \begin{cases} \Lambda_\beta(c), & \\ \Lambda_{\beta\{\|y_4\|^s+\|y_3\|^s+\|y_2\|^s+\|y_1\|^s\}}(c), & s \neq 1\,; \\ \Lambda_{\beta\{\|y_4\|^{4s}+\|y_3\|^{4s}+\|y_2\|^{4s}+\|y_1\|^{4s}+\|y_4\|^s\|y_3\|^s\|y_2\|^s\|y_1\|^s\}}(c), & 4\,s \neq 1\,; \end{cases} \qquad (4.24)$$

for all $y_4, y_3, y_2, y_1 \in K$ and all $c > 0$. Then there exists a unique additive function $A : K \to L$such that

$$\Xi_{P_a(y)-A(y)}(c) \geq \begin{cases} \Lambda_{\frac{5\beta}{9}}(c), \\ \Lambda_{\frac{5\beta\|y\|^s\left(8+\frac{1}{4^s}+\frac{1}{3^s}\right)}{4|10-10^s|}}(c), \\ \Lambda_{\frac{5\beta\|y\|^{4s}}{4|10-10^{4s}|}}(c), \end{cases} \tag{4.25}$$

for all $y \in K$ and all $c > 0$.

Theorem 4.3. Assume that $u \in \{\pm 1\}$. Let K be a linear space, (L, Ξ, T) be a complete RN-space and $P_q : K \to L$ be an even mapping for which there exist a function $\Lambda : \mathrm{K}^4 \to \mathrm{D}^+$with the inequality

$$\Xi_{GP_q(y_4,y_3,y_2,y_1)}(c) \geq \Lambda_{y_4,y_3,y_2,y_1}(c) \tag{4.26}$$

for all $y_4, y_3, y_2, y_1 \in K$ and all $c > 0$. If

$$\lim_{m\to\infty} T_{l=0}^{\infty} \Lambda_{10^{ul}y,10^{ul}y,10^{ul}y,10^{ul}y}\left(10^{2u(l+1)}c\right) = 1 = \lim_{l\to\infty} \Lambda_{10^{ul}y_4,10^{ul}y_3,10^{ul}y_2,10^{ul}y_1}\left(10^{2ul}c\right) \tag{4.27}$$

for all $y_4, y_3, y_2, y_1 \in K$ and all $c > 0$. Then there exists a unique quadratic mapping $Q : K \to L$such that

$$\Xi_{Q(y)-P_q(y)}(12c) \geq T_{l=0}^{\infty} \Lambda^{q}_{10^{ul}y}\left(10^{2u(l+1)}c\right) \tag{4.28}$$

for all $y_4, y_3, y_2, y_1 \in K$ and all $c > 0$.

Proof: Assume $u = 1$. Substituting (y_4, y_3, y_2, y_1) with (y, y, y, y) in (4.26) and apply evenness, we get

$$\Xi_{P_q(10y)+P_q(8y)+P_q(6y)+2P_q(4y)-78P_q(2y)+80P_q(y)}(c) \geq \Lambda_{y,y,y,y}(c) \tag{4.29}$$

for all $y \in K$ and all $c > 0$. Again Substituting (y_4, y_3, y_2, y_1) with $(y, 0, y, 2y)$ in (4.26)and apply evenness, we obtain

$$\Xi_{P_q(8y)+2P_q(4y)-2P_q(3y)-78P_q(y)}\left(\frac{c}{2}\right) \geq \Lambda_{y,0,y,2y}(c) \tag{4.30}$$

for all $y \in K$ and all $c > 0$. Again Substituting (y_4, y_3, y_2, y_1) with $(0,0, y, y)$ in (4.26) and apply evenness, we get

$$\Xi_{2P_q(3y)-P_q(2y)-14P_q(y)}\left(\frac{c}{2}\right) \geq \Lambda_{0,0,y,y}(c) \tag{4.31}$$

for all $y \in K$ and all $c > 0$. Again Substituting (y_4, y_3, y_2, y_1) with $(y, 0, y, 0)$ in (4.26) and apply evenness, we get

$$\Xi_{P_q(6y)+P_q(2y)-40P_q(y)}\left(\frac{c}{4}\right) \geq \Lambda_{y,0,y,0}(c) \tag{4.32}$$

for all $y \in K$ and all $c > 0$. Again Substituting (y_4, y_3, y_2, y_1) with $(0,0, y, 0)$ in (4.26) and apply evenness, we get

$$\Xi_{P_q(2y)-4P_q(y)}\left(\frac{c}{8}\right) \geq \Lambda_{(0,0,y,0)}(c) \tag{4.33}$$

for all $y \in K$ and all $c > 0$. It follows from (4.33) that

$$\Xi_{77P_q(2y)-308P_q(y)}\left(\frac{77c}{8}\right) \geq \Lambda_{(0,0,y,0)}(c) \tag{4.34}$$

for all $y \in K$ and all $c > 0$. It follows from (4.32) and (4.33), we acquire

$$\Xi_{P_q(6y)-36P_q(y)}\left(\frac{3c}{8}\right) \geq T\left(\Lambda_{y,0,y,0}(c), \Lambda_{0,0,y,0}(c)\right) \tag{4.35}$$

for all $y \in K$ and all $c > 0$. It follows from (4.30) and (4.31), we obtain

$$\Xi_{P_q(8y)+2P_q(4y)-P_q(2y)-92P_q(y)}(c) \geq T\left(\Lambda_{y,0,y,2y}(c), \Lambda_{0,0,y,y}(c)\right) \tag{4.36}$$

for all $y \in K$ and all $c > 0$. It follows from (4.29) and (4.36), we arrive

$$\Xi_{P_q(10y)+P_q(6y)-77P_q(2y)+172P_q(y)}(2c) \geq \left(T\left(T\left(\Lambda_{y,0,y,2y}(c), \Lambda_{0,0,y,y}(c)\right), \Lambda_{y,y,y,y}(c)\right)\right) \quad (4.37)$$

for all $y \in K$ and all $c > 0$. It follows from (4.34) and (4.37), we acquire

$$\Xi_{P_q(10y)+P_q(6y)-136P_q(y)}\left(\frac{93c}{8}\right) \geq T\left(T\left(T\left(\Lambda_{y,0,y,2y}(c), \Lambda_{0,0,y,y}(c)\right), \Lambda_{y,y,y,y}(c)\right), \Lambda_{0,0,y,0}(c)\right) \quad (4.38)$$

for all $y \in K$ and all $c > 0$. It follows from (4.35) and (4.38), we achieve

$$\Xi_{P_q(10y)-100P_q(y)}(12c) \geq T\left(T\left(T\left(T\left(\Lambda_{y,0,y,2y}(c), \Lambda_{0,0,y,y}(c)\right), \Lambda_{y,y,y,y}(c)\right), \Lambda_{0,0,y,0}(c)\right), T\left(\Lambda_{y,0,y,0}(c), \Lambda_{0,0,y,0}(c)\right)\right) \quad (4.39)$$

for all $y \in K$ and all $c > 0$. It follows from (4.39), we acquire

$$\Xi_{P_q(10y)-10^2P_q(y)}(12c) \geq \Lambda_y^q(c) \quad (4.40)$$

for all $y \in K$ and all $c > 0$. where

$$\Lambda_y^q(c) = T\left(T\left(T\left(T\left(\Lambda_{y,0,y,2y}(c), \Lambda_{0,0,y,y}(c)\right), \Lambda_{y,y,y,y}(c)\right), \Lambda_{0,0,y,0}(c)\right), T\left(\Lambda_{y,0,y,0}(c), \Lambda_{0,0,y,0}(c)\right)\right)$$

for all $y \in K$ and all $c > 0$. It follows from (4.40) and $(RN2)$, we obtain

$$\Xi_{\frac{P_q(10y)}{10^2}-P_q(y)}(12c) \geq \Lambda_y^q(10^2c) \quad (4.41)$$

for all $y \in K$ and all $c > 0$. The rest of the proof is similar to that Theorem 4.1. for $u = 1$. For $u = -1$, Plugging $y = \frac{y}{10}$ in (4.40), we achieve

$$\Xi_{P_q\left(\frac{y}{10}\right)-10^2P_q\left(\frac{y}{10^2}\right)}(5c) \geq \Lambda_{\frac{y}{10}}^q(c) \quad (4.42)$$

for all $y \in K$ and all $c > 0$. The rest of the proof is similar to that Theorem 4.1. This completes the proof of the theorem.
The following Corollary is an immediate consequence of Theorem 4.3 concerning the stability of (1.8).

Corollary 4.4. Let β and s be nonnegative real numbers. Let an even function $P_q : K \to L$satisfies the inequality

$$\Xi_{GP_q(y_4,y_3,y_2,y_1)}(c) \geq \begin{cases} \Lambda_\beta(c), & \\ \Lambda_{\beta\{\|y_4\|^s+\|y_3\|^s+\|y_2\|^s+\|y_1\|^s\}}(c), & s \neq 2; \\ \Lambda_{\beta\{\|y_4\|^{4s}+\|y_3\|^{4s}+\|y_2\|^{4s}+\|y_1\|^{4s}+\|y_4\|^s\|y_3\|^s\|y_2\|^s\|y_1\|^s\}}(c), & 4s \neq 2; \end{cases} \quad (4.43)$$

for all $y_4, y_3, y_2, y_1 \in K$ and all $c > 0$. Then there exists a unique quadratic function $Q : K \to L$ such that

$$\Xi_{P_q(y)-Q(y)}(c) \geq \begin{cases} \Lambda_{\frac{12\beta}{99}}(c), \\ \Lambda_{\frac{2\beta\|y\|^s(2^s+12)}{|10-10^{2s}|}}(c), \\ \Lambda_{\frac{2\beta\|y\|^{4s}}{|10-10^{4s}|}}(c), \end{cases} \quad (4.44)$$

for all $y \in K$ and all $c > 0$.

Theorem 4.5. Let $u \in \{\pm 1\}$. Let $P : K \to L$ be a mapping for which there exist a function $\Lambda : \mathrm{K}^4 \to \mathrm{D}^+$ with the conditions (4.2) and (4.27) such that the functional inequality

$$\Xi_{GP(y_4,y_3,y_2,y_1)}(c) \geq \Lambda_{y_4,y_3,y_2,y_1}(c) \quad (4.45)$$

for all $y_4, y_3, y_2, y_1 \in K$ and all $c > 0$. Then, there exists a unique additive mapping

$A : K \to L$ and a unique quadratic mapping $Q : K \to L$ satisfying the functional equation (1.8) and

$$\Xi_{A(y)-Q(y)-P(y)}(34c) \geq T\left\{T\left(T_{l=0}^{\infty}\Lambda^{a}_{10^{ul}y}\left(10^{(l+1)u}c\right), T_{l=0}^{\infty}\Lambda^{a}_{-10^{ul}y}\left(10^{(l+1)u}c\right)\right),\right.$$
$$\left. T\left(T_{l=0}^{\infty}\Lambda^{q}_{10^{ul}y}\left(10^{2(l+1)u}c\right), T_{l=0}^{\infty}\Lambda^{q}_{-10^{ul}y}\left(10^{2(l+1)u}c\right)\right)\right\} \tag{4.46}$$

for all $y \in K$ and all $c > 0$.

Proof: Let $P_a(y) = \frac{P(y)-P(-y)}{2}$ for all $y \in K$. Then $P_a(0) = 0$ and $P_a(-y) = -P_a(y)$ for all $y \in K$. Hence

$$\Xi_{GP_a(y_4,y_3,y_2,y_1)}(2c) \geq T\left(\Lambda_{y_4,y_3,y_2,y_1}(c), \Lambda_{-y_4,-y_3,-y_2,-y_1}(c)\right) \tag{4.47}$$

for all $y_4, y_3, y_2, y_1 \in K$ and all $c > 0$. By Theorem 4.1, we have

$$\Xi_{A(y)-P_a(y)}(10c) \geq T_{l=0}^{\infty}\left(\Lambda^{a}_{10^{ul}y}\left(10^{u(l+1)}c\right), \Lambda^{a}_{-10^{ul}y}\left(10^{u(l+1)}c\right)\right) \tag{4.48}$$

for all $y \in K$ and all $c > 0$. Also, let $P_q(y) = \frac{P(y)+P(-y)}{2}$ for all $y \in K$ and all $c > 0$. Then $P_q(0) = 0$ and $P_q(-y) = P_q(y)$ for all $y \in K$ and all $c > 0$. Hence

$$\Xi_{GP_q(y_4,y_3,y_2,y_1)}(2c) \geq T\left(\Lambda_{y_4,y_3,y_2,y_1}(c), \Lambda_{-y_4,-y_3,-y_2,-y_1}(c)\right) \tag{4.49}$$

for all $y \in K$ and all $c > 0$. By Theorem 4.3, we have

$$\Xi_{Q(y)-P_q(y)}(24c) \geq T_{l=0}^{\infty}\left(\Lambda^{q}_{10^{ul}y}\left(10^{2u(l+1)}c\right), \Lambda^{q}_{-10^{ul}y}\left(10^{2u(l+1)}c\right)\right) \tag{4.50}$$

for all $y \in K$ and all $c > 0$. Define

$$P(y) = P_a(y) + P_q(y) \tag{4.51}$$

for all $y \in K$ and all $c > 0$. From (4.48), (4.50) and (4.51), we arrive

$$\begin{aligned}\Xi_{A(y)-Q(y)-P(y)}(34c) &= \mu_{A(y)-Q(y)-P_a(y)-P_q(y)}(34c)\\ &\geq T\left(\Xi_{A(y)-P_a(y)}(10c), \Xi_{Q(y)-P_q(y)}(24c)\right)\\ &\geq T\left\{T\left(T_{l=0}^{\infty}\Lambda^{a}_{10^{ul}y}\left(10^{u(l+1)}c\right), T_{l=0}^{\infty}\Lambda^{a}_{-10^{ul}y}\left(10^{u(l+1)}c\right)\right),\right.\\ &\qquad \left. T\left(T_{l=0}^{\infty}\Lambda^{q}_{10^{ul}y}\left(10^{2(l+1)u}c\right), T_{l=0}^{\infty}\Lambda^{q}_{-10^{ul}y}\left(10^{2(l+1)u}c\right)\right)\right\}\end{aligned}$$

for all $y \in K$ and all $c > 0$. Hence the theorem is proved.
Using Corollaries 4.2 and 4.4, we have the following corollary concerning the stability of (1.8).

Corollary 4.6. Let β and l be nonnegative real numbers. Let a function $P : K \to L$ satisfies the inequality

$$\Xi_{GP(y_4,y_3,y_2,y_1)}(c) \geq \begin{cases} \Lambda_{\beta}(c), & \\ \Lambda_{\beta\{\|y_4\|^s+\|y_3\|^s+\|y_2\|^s+\|y_1\|^s\}}(c), & s \neq 1,2; \\ \Lambda_{\beta\{\|y_4\|^{4s}+\|y_3\|^{4s}+\|y_2\|^{4s}+\|y_1\|^{4s}+\|y_4\|^s\|y_3\|^s\|y_2\|^s\|y_1\|^s\}}(c), & 4s \neq 1,2; \end{cases} \tag{4.52}$$

for all $y_4, y_3, y_2, y_1 \in K$ and all $c > 0$. Then there exists a unique additive function $A : K \to L$ and a unique quadratic function $Q : K \to L$ such that

$$\Xi_{A(y)-Q(y)-P(y)}(34c) \geq \begin{cases} \Lambda_{\frac{5\beta}{9}+\frac{12\beta}{99}}(c), \\ \Lambda_{\beta\|y\|^s\left(\frac{5\left(8+\frac{1}{4^s}+\frac{1}{3^s}\right)}{4|10-10^s|}+\frac{2(2^s+12)}{|10-10^{2s}|}\right)}(c), \\ \Lambda_{\beta\|y\|^{4s}\left(\frac{5}{4|10-10^{4s}|}+\frac{2}{|10-10^{4s}|}\right)}(c), \end{cases} \tag{4.53}$$

for all $y \in K$ and all $c > 0$.

4.2. Fixed Point Method. In this section, the authors prove the generalized Ulam-Hyers stability of functional equation (1.8) in RN-space with the help of the fixed point method. Now, we will revise the fundamental results in fixed point theory.

Theorem 4.8. (The alternative of fixed point) [24]

Suppose that for a complete generalized metric space (X, d) and a strictly contractive mapping $T: X \to X$ with Lipschitz constant L. Then, for each given element $x \in X$, either

$(B1)$ $d\,(T^n x, T^{n+1}x) = \infty \ \ \forall\, n \geq 0$,

Or

$(B2)$ there exists a natural number n_0 such that:

i) $d(T^n x, T^{n+1}x) < \infty \ \ \forall\, n \geq 0$;

ii) The sequence$(T^n x)$ is convergent to a fixed pointy^* of T

iii) $y^*$is the unique fixed point of T in the set

$$\Upsilon = \{y \in X : d\,(T^{n_0}x, y) < \infty\};$$

iv) $d(y^*, y) \leq \frac{1}{1-L}\, d\big(y, T_y\big)$ for all $y \in \Upsilon$.

Theorem 4.9. Let $P_a : K \to L$ be an odd mapping for which there exists a function $\Lambda : K^4 \to D^+$with the condition

$$\lim_{l\to\infty} \Lambda_{\omega_b^l t_4, \omega_b^l t_3, \omega_b^l t_2, \omega_b^l t_1}\big(\omega_b^l c\big) = 1 \tag{4.54}$$

where $\omega_b = 10$ if $b = 0$ and $\omega_b = \frac{1}{10}$ if $b = 1$ such that the functional inequality with

$$\Xi_{GP_a(y_4,y_3,y_2,y_1)}(c) \geq \Lambda_{y_4,y_3,y_2,y_1}(c) \tag{4.55}$$

for all $y_4, y_3, y_2, y_1 \in K$ and all $c > 0$. If there exists $L = L(b)$ such that the function

$$\Lambda^a_{\frac{y}{10}}(c) = \Lambda^a_y(c)$$

has the property

$$\Lambda^a_{\omega_b y}(\omega_b c) = \Lambda^a_y(Lc) \tag{4.56}$$

for all $y \in K$ and all $c > 0$. Then, there exists a unique additive function $A : K \to L$ satisfying the functional equation (1.8) and

$$\Xi_{A(y)-P_a(y)}\left(\frac{L^{1-b}}{1-L}\,5c\right) \geq \Lambda^a_y(c) \tag{4.57}$$

holds for all $y \in K$ and all $c > 0$.

Proof: Consider the set $H = \{h/h : K \to L, h(0) = 0\}$ and introduce the generalized metric on H by

$$d(h_1, h_2) = inf\big\{K \in (0,\infty) : \Xi_{h_1(y)-h_2(y)}(Kc) \geq \Lambda^a_y(c), y \in K, c > 0\big\}$$

It is easy to see that (H, d) is complete. Define $G : H \to H$ by

$$Gh_1(y) = \frac{1}{\omega_b} h_1(\omega_b y)$$

for all $y \in K$ and all $c > 0$. Now $h_1, h_2 \in H$,

$$d(h_1, h_2) \leq K \quad \Rightarrow \Xi_{h_1(y)-h_2(y)}(Kc) \geq \Lambda^a_y(c), y \in K\ and\ c > 0$$

$$\Rightarrow \Xi_{\frac{1}{\omega_b}h_1(\omega_b y)-\frac{1}{\omega_b}h_2(\omega_b y)}\left(K\frac{c}{\omega_b}\right) \geq \Lambda^a_{\omega_b y}(c), y \in K\ and\ c > 0$$

$$\Rightarrow \Xi_{Gh_1(y)-Gh_2(y)}(KLc) \geq \Lambda^a_y(c), y \in K\ and\ c > 0$$

$$d(Gh_1, Gh_2) \leq KL$$

This implies $d(Gh_1, Gh_2) \leq Kd(h_1, h_2)$, for all $h_1, h_2 \in H$.

i.e., G is a strictly contractive mapping on H with Lipschitz constant $L = \frac{1}{\omega_b}$.

It follows that (4.12), we acqire

$$\Xi_{\frac{P_a(10y)}{10}-P_a(y)}(5c) \geq \Lambda^a_y(10c) \tag{4.58}$$

for all $y \in K$ and all $c > 0$. Using (4.56) for the case $b = 0$ it reduces to

$$\Xi_{GP_a(y)-P_a(y)}(5c) \geq \Lambda_y^a(c)$$

for all $y \in K$ and all $c > 0$.

$$d(GP_a, P_a) \leq L = L^1 < \infty \tag{4.59}$$

Substituting y by $\frac{y}{10}$ in (4.12), we achieve

$$\Xi_{P_a(y)-10P_a(\frac{y}{10})}(5c) \geq \Lambda_{\frac{y}{10}}^a(c)$$

for all $y \in K$ and all $c > 0$. Using (4.56) for the case $b = 1$ it reduces to

$$\Xi_{P_a(y)-GP_a(y)}(5c) \geq \Lambda_y^a(c)$$

for all $y \in K$ and all $c > 0$.

$$d(P_a, GP_a) \leq 1 = L^0 < \infty \tag{4.60}$$

From (4.59) and (4.60), we achieve

$$d(P_a, GP_a) \leq L^{1-b} \tag{4.61}$$

Therefore, in both cases $B2(i)$ holds.
By $B2(ii)$, it follows that there exists a fixed point A of G in H such that

$$\Xi_{A(y)}(c) = \lim_{l\to\infty} \Xi_{\frac{P_a(\omega_b^l y)}{\omega_b^l}}(c) \tag{4.62}$$

for all $y \in K$ and all $c > 0$.
In order to prove $A : K \to L$ is Additive, we are replacing (y_4, y_3, y_2, y_1) by $\left(\omega_b^l y_4, \omega_b^l y_3, \omega_b^l y_2, \omega_b^l y_1\right)$in(4.55), It follows from (4.62) and (4.62), we acquire A satisfies (1.8).
By $B2(iii)$, A is the unique fixed point of G in the set $\Delta = \{P_a \in C: d(P_a, A) < \infty\}$, using the fixed point alternative result A is the unique function such that

$$\Xi_{P_a(y)-A(y)}(5Kc) \geq \Lambda_y^a(c)$$

for all $y \in K$ and all $c > 0$. Again by $B2(iv)$, we obtain

$$d(P_a, A) \leq \frac{1}{1-L} d(P_a, GP_a)$$

which implies

$$d(P_a, A) \leq \frac{L^{1-b}}{1-L}$$

Hence, we conclude that

$$\Xi_{P_a(y)-A(y)}\left(\frac{L^{1-b}}{1-L}\ 5c\right) \geq \Lambda_y^a(c)$$

for all $y \in K$ and all $c > 0$. This completes the proof of the theorem.
The following corollary is an immediate consequence of Theorem 4.9 concerning the stability of (1.8).

Corollary 4.10. Let β and s be nonnegative real numbers. Let an odd function $P_a : K \to L$satisfies the inequality

$$\Xi_{GP_a(y_4,y_3,y_2,y_1)}(c) \geq \begin{cases} \Lambda_\beta(c), & \\ \Lambda_{\beta\{\|y_4\|^s+\|y_3\|^s+\|y_2\|^s+\|y_1\|^s\}}(c), & s \neq 1\,; \\ \Lambda_{\beta\{\|y_4\|^{4s}+\|y_3\|^{4s}+\|y_2\|^{4s}+\|y_1\|^{4s}+\|y_4\|^s\|y_3\|^s\|y_2\|^s\|y_1\|^s\}}(c), & 4\,s \neq 1\,; \end{cases} \tag{4.63}$$

for all $y_4, y_3, y_2, y_1 \in K$ and all $c > 0$. Then there exists a unique additive function $A : K \to L$ such that

$$\Xi_{P_a(y)-A(y)}(c) \geq \begin{cases} \Lambda_{\frac{5\beta}{9}}(c), \\ \Lambda_{\frac{5\beta\|y\|^s\left(8+\frac{1}{4^s}+\frac{1}{3^s}\right)}{4|10-10^s|}}(c), \\ \Lambda_{\frac{5\beta\|y\|^{4s}}{4|10-10^{4s}|}}(c), \end{cases} \tag{4.64}$$

for all $y \in K$ and all $c > 0$.

Proof: Let us setting

$$\Lambda_{y_4,y_3,y_2,y_1}(c)=\begin{cases}\Lambda_{\beta}(c),\\ \Lambda_{\beta\{\|y_4\|^s+\|y_3\|^s+\|y_2\|^s+\|y_1\|^s\}}(c),\\ \Lambda_{\beta\{\|y_4\|^s\|y_3\|^s\|y_2\|^s\|y_1\|^s\}}(c),\end{cases}$$

for all $y\in K$ and all $c>0$. Then

$$\Lambda_{\frac{1}{\omega_b^l}\omega_b^l y_4,\omega_b^l y_3,\omega_b^l y_2,\omega_b^l y_1}\left(\omega_b^l c\right)=\begin{cases}\Lambda_{\frac{\beta}{\omega_b^l}}(c),\\ \Lambda_{\frac{\beta}{\omega_b^l}\{\|\omega_b^l y_4\|^s+\|\omega_b^l y_3\|^s+\|\omega_b^l y_2\|^s+\|\omega_b^l y_1\|^s\}}(c),\\ \Lambda_{\frac{\beta}{\omega_b^l}\{\|\omega_b^l y_4\|^s\|\omega_b^l y_3\|^s\|\omega_b^l y_2\|^s\|\omega_b^l y_1\|^s\}}(c),\end{cases}$$

$$=\begin{cases}\to 1\ as\ l\to\infty,\\ \to 1\ as\ l\to\infty,\\ \to 1\ as\ l\to\infty,\end{cases}$$

Then (4.62) holds. Now,

$$\Lambda_y^a(c)=\begin{cases}\Lambda_{\beta}(c),\\ \Lambda_{\frac{\beta\|y\|^s\left(8+\frac{1}{4^s}+\frac{1}{3^s}\right)}{4\cdot 10^s}}(c),\\ \Lambda_{\frac{\beta\|y\|^{4s}}{4\cdot 10^{4s}}}(c),\end{cases}$$

$$\Lambda_{\frac{y}{\omega_b}}^a\left(\omega_b^{-1}c\right)=\begin{cases}\Lambda_{\beta}\left(\omega_b^{-1}c\right),\\ \Lambda_{\frac{\beta\|y\|^s}{\omega_b^s}}\left(\omega_b^{-1}c\right),\\ \Lambda_{\frac{\beta\|y\|^{4s}}{\omega_b^{4s}}}\left(\omega_b^{-1}c\right),\end{cases}$$

$$\Lambda_{\frac{y}{\omega_b}}^a\left(\omega_b^{-1}c\right)=\begin{cases}\Lambda_{\beta}\left(\omega_b^{-1}c\right),\\ \Lambda_{\beta\|y\|^s}\left(\omega_b^{s-1}c\right),\\ \Lambda_{\beta\|y\|^{4s}}\left(\omega_b^{4s-1}c\right),\end{cases}$$

(4.65) can be rewritten as

$$\Xi_{P_a(y)-A(y)}(5c)\geq\Lambda_y^a\left(\frac{1-L}{L^{1-b}}\ c\right)$$

Case:1 $L=10^{-1}$ for $s=1$ if $b=0$,

$$\begin{aligned}\Xi_{P_a(y)-A(y)}(5c)&\geq\Lambda_y^a\left(\frac{1-L}{L^{1-b}}\ c\right)\\ &=\Lambda_y^a\left(\frac{1-10^{-1}}{(10^{-1})^{1-0}}\ c\right)\\ &=\Lambda_y^a(9c)\end{aligned}$$

$$\Xi_{P_a(y)-A(y)}\left(\frac{5}{9}c\right)=\Lambda_y^a(c)$$

Case:2 $L=10$ for $s=0$ if $b=1$,

$$\begin{aligned}\Xi_{P_a(y)-A(y)}(5c)&\geq\Lambda_y^a\left(\frac{1-L}{L^{1-b}}\ c\right)\\ &=\Lambda_y^a\left(\frac{1-\frac{1}{10^{-1}}}{\left(\frac{1}{10^{-1}}\right)^{1-1}}\ c\right)\\ &=\Lambda_y^a\big((1-10)c\big)\end{aligned}$$

$$\Xi_{P_a(y)-A(y)}\left(-\frac{5}{9}c\right)=\Lambda_y^a(c)$$

Case:3 $L = 10^{s-1}$ for $s < 1$ if $b = 0$,

$$\Xi_{P_a(y)-A(y)}(5c) \geq \Lambda^a_{\frac{\beta\|y\|^s\left(8+\frac{1}{4^s}+\frac{1}{3^s}\right)}{4}}\left(\frac{1-L}{L^{1-b}}\ c\right)$$

$$= \Lambda^a_{\frac{\beta\|y\|^s\left(8+\frac{1}{4^s}+\frac{1}{3^s}\right)}{4}}\left(\frac{1-10^{s-1}}{(10^{s-1})^{1-0}}\ c\right)$$

$$= \Lambda^a_{\frac{\beta\|y\|^s\left(8+\frac{1}{4^s}+\frac{1}{3^s}\right)}{4}}\left(\frac{10-10^s}{10^s}\ c\right)$$

$$\Xi_{P_a(y)-A(y)}\left(\frac{10^s}{10-10^s}5c\right) = \Lambda^a_{\frac{\beta\|y\|^s\left(8+\frac{1}{4^s}+\frac{1}{3^s}\right)}{4}}(\ c)$$

Case:4 $L = \frac{1}{10^{s-1}}$ for $s > 1$ if $b = 1$,

$$\Xi_{P_a(y)-A(y)}(5c) \geq \Lambda^a_{\frac{\beta\|y\|^s\left(8+\frac{1}{4^s}+\frac{1}{3^s}\right)}{4}}\left(\frac{1-\frac{1}{10^{s-1}}}{\left(\frac{1}{10^{s-1}}\right)^{1-1}}\ c\right)$$

$$= \Lambda^a_{\frac{\beta\|y\|^s\left(8+\frac{1}{4^s}+\frac{1}{3^s}\right)}{4}}\left(\frac{10^s-10}{10^s}\ c\right)$$

$$\Xi_{P_a(y)-A(y)}\left(\frac{10^s}{10^s-10}5c\right) = \Lambda^a_{\frac{\beta\|y\|^s\left(8+\frac{1}{4^s}+\frac{1}{3^s}\right)}{4}}(\ c)$$

Case:5 $L = 10^{4s-1}$ for $s < \frac{1}{4}$ if $b = 0$,

$$\Xi_{P_a(y)-A(y)}(5c) \geq \Lambda^a_{\frac{\beta\|y\|^{4s}}{4}}\left(\frac{1-L}{L^{1-b}}\ c\right)$$

$$= \Lambda^a_{\frac{\beta\|y\|^{4s}}{4}}\left(\frac{1-10^{4s-1}}{(10^{4s-1})^{1-0}}\ c\right)$$

$$= \Lambda^a_{\frac{\beta\|y\|^{4s}}{4}}\left(\frac{10-10^{4s}}{10^{4s}}\ c\right)$$

$$\Xi_{P_a(y)-A(y)}\left(\frac{10^{4s}}{10-10^{4s}}5c\right) = \Lambda^a_{\frac{\beta\|y\|^{4s}}{4}}(\ c)$$

Case:6 $L = \frac{1}{10^{4s-1}}$ for $s > \frac{1}{4}$ if $b = 1$,

$$\Xi_{P_a(y)-A(y)}(5c) \geq \Lambda^a_{\frac{\beta\|y\|^{4s}}{4}}\left(\frac{1-L}{L^{1-b}}\ c\right)$$

$$= \Lambda^a_{\frac{\beta\|y\|^{4s}}{4}}\left(\frac{1-\frac{1}{10^{4s-1}}}{\left(\frac{1}{10^{4s-1}}\right)^{1-1}}\ c\right)$$

$$= \Lambda^a_{\frac{\beta\|y\|^{4s}}{4}}\left(\frac{10^{4s}-10}{10^{4s}}\ c\right)$$

$$\Xi_{P_a(y)-A(y)}\left(\frac{10^{4s}}{10^{4s}-10}5c\right) = \Lambda^a_{\frac{\beta\|y\|^{4s}}{4}}(\ c)$$

Hence the proof is complete.

Theorem 4.11. Let $P_q : K \to L$ be an even mapping for which there exists a function $\Lambda : K^4 \to D^+$with the condition

$$\lim_{l\to\infty} \Lambda_{\omega_b^{2l}t_4,\omega_b^{2l}t_3\omega_b^{2l}t_2,\omega_b^{2l}t_1}\left(\omega_b^{2l}c\right) = 1 \quad (4.65)$$

where $\omega_b = 10$ if $b = 0$ and $\omega_b = \frac{1}{10}$ if $b = 1$ such that the functional inequality with

$$\Xi_{GP_q(y_4,y_3,y_2,y_1)}(c) \geq \Lambda_{y_4,y_3,y_2,y_1}(c) \qquad (4.66)$$

for all $y_4, y_3, y_2, y_1 \in K$ and all $c > 0$. If there exists $L = L(b)$ such that the function

$$\Lambda^q_{\frac{y}{10}}(c) = \Lambda^q_y(c)$$

has the property

$$\Lambda^q_{\omega_b{}^2 y}(\omega_b{}^2 c) = \Lambda^q_y(L^2 c) \qquad (4.67)$$

for all $y \in K$ and all $c > 0$. Then, there exists a unique Quadratic function $Q : K \to L$ satisfying the functional equation (1.8) and

$$\Xi_{Q(y)-P_q(y)}\left(\frac{L^{1-b}}{1-L}\ 12c\right) \geq \Lambda^q_y(c) \qquad (4.68)$$

holds for all $y \in K$ and all $c > 0$.

Proof: Consider the set $H = \{h/h : K \to L, h(0) = 0\}$ and introduce the generalized metric on H by

$$d(h_1, h_2) = inf\left\{K \in (0,\infty) : \Xi_{h_1(y)-h_2(y)}(Kc) \geq \Lambda^q_y(c), y \in K, c > 0\right\}$$

It is easy to see that (H, d) is complete. Define $G : H \to H$ by

$$Gh_1(y) = \frac{1}{\omega_b^2} h_1\left(\omega_b^2 y\right)$$

for all $y \in K$ and all $c > 0$. Now $h_1, h_2 \in H$,

$$d(h_1, h_2) \leq K \Rightarrow \Xi_{h_1(y)-h_2(y)}(Kc) \geq \Lambda^q_y(c), y \in K\ and\ c > 0$$

$$\Rightarrow \Xi_{\frac{1}{\omega_b^2}h_1(\omega_b^2 y)-\frac{1}{\omega_b^2}h_2(\omega_b^2 y)}\left(K\frac{c}{\omega_b^2}\right) \geq \Lambda^q_y(c), y \in K\ and\ c > 0$$

$$\Rightarrow \Xi_{Gh_1(y)-Gh_2(y)}(KLc) \geq \Lambda^q_y(c), y \in K\ and\ c > 0$$

$$d(Gh_1, Gh_2) \leq KL$$

This implies $d(Gh_1, Gh_2) \leq \omega_b^2 d(h_1, h_2)$, for all $h_1, h_2 \in H$.

i.e., G is a strictly contractive mapping on H with Lipschitz constant $L = \frac{1}{\omega_b^2}$.

It follows that (4.40), we acqire

$$\Xi_{\frac{P_q(10y)}{10^2}-P_q(y)}(12c) \geq \Lambda^q_y(10^2 c) \qquad (4.69)$$

for all $y \in K$ and all $c > 0$. Using (4.67) for the case $b = 0$ it reduces to

$$\Xi_{GP_q(y)-P_q(y)}(12c) \geq \Lambda^q_y(c)$$

for all $y \in K$ and all $c > 0$.

$$d\left(GP_q, P_q\right) \leq L = L^1 < \infty \qquad (4.70)$$

Substituting y by $\frac{y}{10}$ in (4.40), we achieve

$$\Xi_{P_q(y)-10^2 P_q(\frac{y}{10})}(12c) \geq \Lambda^q_{\frac{y}{10}}(c)$$

for all $y \in K$ and all $c > 0$. Using (4.67) for the case $b = 1$ it reduces to

$$\Xi_{P_q(y)-GP_q(y)}(12c) \geq \Lambda^q_y(c)$$

for all $y \in K$ and all $c > 0$.

$$d\left(P_q, GP_q\right) \leq 1 = L^0 < \infty \qquad (4.71)$$

From (4.70) and (4.71), we achieve

$$d\left(P_q, GP_q\right) \leq L^{1-b} \qquad (4.72)$$

Therefore, in both cases $B2(i)$ holds.

By $B2(ii)$, it follows that there exists a fixed point Q of G in H such that

$$\Xi_{Q(y)}(c) = \lim_{l\to\infty} \Xi_{\frac{P_q(\omega_b^{2l} y)}{\omega_b^{2l}}}(c) \qquad (4.73)$$

for all $y \in K$ and all $c > 0$. In order to prove $Q : K \to L$ is Quadratic, we are replacing (y_4, y_3, y_2, y_1) by $\left(\omega_b^{2l} y_4, \omega_b^{2l} y_3, \omega_b^{2l} y_2, \omega_b^{2l} y_1\right)$in(4.66), It follows from (4.65) and (4.73), we acquire Qsatisfies (1.8).

By $B2(iii)$, Q is the unique fixed point of G in the set $\Delta=\{P_q \in C : d(P_q, A) < \infty\}$,
Usingthe fixed point alternative result A is the unique function such that

$$\Xi_{P_q(y)-Q(y)}(12Kc) \geq \Lambda_y^q(c)$$

for all $y \in K$ and all $c > 0$. Again by $B2(iv)$, we obtain

$$d(P_q, Q) \leq \frac{1}{1-L} d(P_q, GP_q)$$

which implies

$$d(P_q, Q) \leq \frac{L^{1-b}}{1-L}$$

Hence, we conclude that

$$\Xi_{P_q(y)-Q(y)}\left(\frac{L^{1-b}}{1-L}\ 12c\right) \geq \Lambda_y^q(c)$$

for all $y \in K$ and all $c > 0$. This completes the proof of the theorem.
The following corollary is an immediate consequence of Theorem 4.11 concerning the stability of (1.8).

Corollary 4.12. Let β and s be nonnegative real numbers. Let an even function $P_q : K \to L$ Satisfies the inequality

$$\Xi_{GP_q(y_4,y_3,y_2,y_1)}(c) \geq \begin{cases} \Lambda_\beta(c), & \\ \Lambda_{\beta\{\|y_4\|^s+\|y_3\|^s+\|y_2\|^s+\|y_1\|^s\}}(c), & s \neq 2; \\ \Lambda_{\beta\{\|y_4\|^{4s}+\|y_3\|^{4s}+\|y_2\|^{4s}+\|y_1\|^s+\|y_4\|^s\|y_3\|^s\|y_2\|^s\|y_1\|^s\}}(c), & 4s \neq 2; \end{cases} \quad (4.74)$$

for all $y_4, y_3, y_2, y_1 \in K$ and all $c > 0$. Then there exists a unique Quadratic function $Q : K \to L$such that

$$\Xi_{P_q(y)-Q(y)}(c) \geq \begin{cases} \Lambda_{\frac{12\beta}{99}}(c), \\ \Lambda_{\frac{2\beta\|y\|^s(2^s+12)}{|10-10^{2s}|}}(c), \\ \Lambda_{\frac{2\beta\|y\|^{4s}}{|10-10^{4s}|}}(c), \end{cases} \quad (4.75)$$

for all $y \in K$ and all $c > 0$.

Proof: The proof is similar to the Corollary 4.10.

Theorem 4.13. Let $P : K \to L$ be a mapping for which there exists a function $\Lambda : K^4 \to D^+$with the condition (4.54) and (4.65) where $w_b = 10$ if $b = 0$ and $w_b = \frac{1}{10}$ if $b = 1$ such that the functional inequality with

$$\Xi_{GP_q(y_4,y_3,y_2,y_1)}(c) \geq \Lambda_{y_4,y_3,y_2,y_1}(c) \quad (4.76)$$

for all $y_4, y_3, y_2, y_1 \in K$ and all $c > 0$. If there exists $L = L(b)$ has the properties (4.56) and (4.67) , then there exists unique additive function $A : K \to L$ a unique quadratic function $Q : K \to L$ satisfying the functional equation (1.8) and

$$\Xi_{A(y)-Q(y)-P(y)}\left(\frac{L^{1-b}}{1-L}\ 34c\right) \geq T\left(T\left(\Lambda_y^a(c), \Lambda_{-y}^a(c)\right), T\left(\Lambda_y^q(c), \Lambda_{-y}^q(c)\right)\right) \quad (4.77)$$

holds for all $y \in K$ and all $c > 0$.

Proof. Let $P_a(y) = \frac{P(y)-P(-y)}{2}$. for all $y \in K$. Then$P_a(0) = 0$ and $P_a(-y) = -P_a(y)$ for all $y \in K$. Hence

$$\Xi_{GP_a(y_4,y_3,y_2,y_1)}(2c) \geq T\left(\Lambda_{y_4,y_3,y_2,y_1}(c), \Lambda_{-y_4,-y_3,-y_2,-y_1}(c)\right) \quad (4.78)$$

for all $y_4, y_3, y_2, y_1 \in K$ and all $c > 0$. By Theorem 4.9, we have

$$\Xi_{A(y)-P_a(y)}\left(\frac{L^{1-b}}{1-L}\ 10c\right) \geq T\left(\Lambda_y^a(c), \Lambda_{-y}^a(c)\right) \quad (4.79)$$

for all $y \in K$ and all $c > 0$.
Also, let $P_q(y) = \frac{P(y)+P(-y)}{2}$ for all $y \in K$. Then $P_q(0) = 0$ and $P(-y) = P(y)$ for all $y \in K$. Hence

$$\Xi_{GP_q(y_4,y_3,y_2,y_1)}(2c) \geq T\left(\Lambda_{y_4,y_3,y_2,y_1}(c), \Lambda_{-y_4,-y_3,-y_2,-y_1}(c)\right) \tag{4.80}$$

for all $y_4, y_3, y_2, y_1 \in K$ and all $c > 0$. By Theorem 4.11, we have

$$\Xi_{Q(y)-P_q(y)}\left(\frac{L^{1-b}}{1-L}\, 24c\right) \geq T\left(\Lambda_y^q(c), \Lambda_{-y}^q(c)\right) \tag{4.81}$$

for all $y \in K$ and all $c > 0$. Define

$$P(y) = P_a(y) + P_q(y) \tag{4.82}$$

for all $y \in K$. From (4.79), (4.81) and (4.82), we arrive

$$\begin{aligned}\Xi_{A(y)-Q(y)-P(y)}\left(\frac{L^{1-b}}{1-L}\, 34c\right) &= \mu_{A(y)-Q(y)-P_a(y)-P_q(y)}\left(\frac{L^{1-b}}{1-L}\, 34c\right) \\ &\geq T\left(\Xi_{A(y)-P_a(y)}\left(\frac{L^{1-b}}{1-L}\, 10c\right), \Xi_{Q(y)-P_q(y)}\left(\frac{L^{1-b}}{1-L}\, 24c\right)\right) \\ &\geq T\left(T\left(\Lambda_y^a(c), \Lambda_{-y}^a(c)\right), T\left(\Lambda_y^q(c), \Lambda_{-y}^q(c)\right)\right)\end{aligned}$$

for all $y \in K$ and all $c > 0$. Hence the theorem is proved.
Using corollaries 4.10 and 4.12, we have the following corollary concerning the stability of (1.8).

Corollary 4.14. Let β and l be nonnegative real numbers. Let a function $P : K \to L$ satisfies the inequality

$$\Xi_{GP(y_4,y_3,y_2,y_1)}(c) \geq \begin{cases} \Lambda_\beta(c), & \\ \Lambda_{\beta\{\|y_4\|^s+\|y_3\|^s+\|y_2\|^s+\|y_1\|^s\}}(c), & s \neq 1,2; \\ \Lambda_{\beta\{\|y_4\|^s+\|y_3\|^s+\|y_2\|^s+\|y_1\|^s+\|y_4\|^s\|y_3\|^s\|y_2\|^s\|y_1\|^s\}}(c), & 4s \neq 1,2; \end{cases} \tag{4.83}$$

for all $y_4, y_3, y_2, y_1 \in K$ and all $c > 0$. Then there exists a unique additive function $A : K \to L$ and a unique quadratic function $Q : K \to L$ such that

$$\Xi_{A(y)-Q(y)-P(y)}(17c) \geq \begin{cases} \Lambda_{\frac{5\beta}{9}+\frac{12\beta}{99}}(c), \\ \Lambda_{\beta\|y\|^s\left(\frac{5\left(8+\frac{1}{4^s}+\frac{1}{3^s}\right)}{4|10-10^s|}+\frac{2(2^s+12)}{|10-10^{2s}|}\right)}(c), \\ \Lambda_{\beta\|y\|^{4s}\left(\frac{5}{4|10-10^{4s}|}+\frac{2}{|10-10^{4s}|}\right)}(c), \end{cases} \tag{4.84}$$

for all $y \in K$ and all $c > 0$.

Acknowledgement
This article is dedicated to all the research scholars and mathematicians those who are working this research field.

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