

s Star p Star Connected Spaces And s Star p Star Compact Spaces in Topological Spaces

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Abstract: The existing article's goal is to establish and explore the new idea of connected and compact spaces which is known as s*p*-connected and s*p*-compact spaces by using s*p*-open sets and s*p*-closed sets. Also, we study and discuss some basic properties and theorems of these topological spaces.

Keywords: s*p*-O set, s*p*-C set, s*p* closure, s*p* connected spaces, s*p*compact spaces.

1. Introduction

In Topological spaces the connected space is defined that it could not be constructed by the joining of two or more open sets that are both non-empty as well as disjoint. Topological characteristics including Connectedness and compactness. The idea of connectedness and compactness are crucial to Topology and other advanced Mathematical fields. Connectivity is a useful tool for proving several theorems and also easily understood. The fundamental characteristics of connectedness and compactness have been extensively studied by scholars. Numerous types of compactness and connectedness have been proposed and examined. The importance of compactness to topology is beyond its relationship to continuous function and definition of compactness follows from the "Heine-Borel" theorem.

Benchalli S.S et al., [1] established gb-connectedness and gb-compactness and semi*connectedness and semi*compactness and also semi* α connectedness and semi* α compactness was derived by Robert A and S.Missier [2, 3]. K.Binoy Balan and C.Janaki [4] proved πp - compact and πp - connected spaces. V.E Sasikala and D. Sivaraj [5] studied semi- open set in soft set in the year 2016. α^* -connected and α^* -compactness in topological spaces was introduced by K. Vithyasangan [6] in the year 2020. Shibani A. M [7] studied rg- compact and connected spaces. D. Sivaraj and V.E. Sasikala [8] studied soft set and α -open sets. αg^*s - connectedness & αg^*s - compactness in topological spaces was introduced by Sarika M. Patil and T.D. Rayanagoudar [9]. K. Vithyasangan et.al.,[10] introduced gr-connectedness in topological spaces. The concepts of Connectedness as well as compactness through semi-star regular open sets were initially explored by S. Pious Missier et.al. [11]. In soft TS, soft swg separation axioms were proposed by V.E. Sasikala and D. Sivaraj [12].

Recently Gnanambal et.al studied and examined GPR-compact, GPR-connected, ω -compact and ω -connectedness in topological spaces. On sg^*b -connected & sg^*b -compactness were established by S. Sekar and B. Jothilakshmi [13]. Soft pre-open set derived by V.E. Sasikala and D. Sivaraj [14] in the year 2017. In weakly topological spaces, the connected and compactness were proposed by Vivekananda Dembre and Pankaj B. Gavali [15].

2. PRELIMINARIES

Definition 2.1: The collection \mathcal{A} of subsets of a space X is said to cover X, or is a covering of X, if X is equal to the sum of the elements of \mathcal{A} . If the elements in it are open subsets of X, then it is termed an open covering of X.

In the present analysis, topological spaces are treated as TS, s^*p^* -closed set as s^*p^* -C set and s^*p^* open set as s^*p^* -O set.

3. s^*p^* CONNECTED SPACE

The concepts of s^*p^* connectedness is defined and some of their characteristics are examine in this paper.

Definition 3.1: Let the TS X is known as s^*p^* -connected, if it could not be able to represented as the union of two disjoint non-empty s^*p^* -O sets.

Example 3.2: Assume that $X = \{r, s, t\}$ $\tau = \{X, \phi, \{r, t\}\}$. s^*p^* -C sets are $\{X, \phi, \{s\}, \{s, t\}, \{r, s\}\}$. s^*p^* -O sets are $\{X, \phi, \{r, t\}, \{r\}, \{t\}\}$. Let $M = \{r\}$ & $N = \{t\}$. Here $M \cup N = \{r, t\} \neq X$. Therefore, it is s^*p^* -connected.

Example 3.3: Consider $X = [-a, 0) \cup (0, a]$ which is the subspaces of \mathbb{R} . Every set in X is non-empty and open. Because all open set is s^*p^* -O set. Then they are disjoint and non-empty s^*p^* -O set and their union is $X = [-a, a]$. Hence it is not s^*p^* -connected.

Example 3.4: The rational \mathbb{Q} are not s^*p^* -connected. If X is a subspace of \mathbb{Q} that contains the points U and V . Assume a number M which is irrational between U and V , and X as the union of the open sets. Since every open set is s^*p^* -O sets $Y \cap (-\infty, \sqrt{M})$ and $Y \cap (\sqrt{M}, \infty)$. Thus, it is a disjoint and non-empty s^*p^* -open sets and their union is X . Hence it is not s^*p^* -connected.

Theorem 3.5: If X is s^*p^* -connected iff the ϕ and X alone are the subsets X that are both s^*p^* -O set as well as s^*p^* -C set.

Proof: Consider P be a proper subset of X . This is non-empty that contains both s^*p^* -O set and s^*p^* -C set. Let $Q = X - P$. Hence P is closed, then $X - P$ is open. Thus, Q is open. since they are s^*p^* -O, disjoint as well as non-empty and their union is X this is contradicting the reality that X is s^*p^* -connected. Therefore, there is no proper subset of X which is both s^*p^* -C & s^*p^* -O sets. Conversely if the only subsets of X are X and ϕ , which are both s^*p^* -O set and s^*p^* -C set. We need to show X is s^*p^* connected. Consider X is not s^*p^* -connected, so $X = P \cup Q$, where P & Q are two nonempty disjoint s^*p^* -O subsets of X which implies that $Q = X - P$, then Q is both s^*p^* -O set and s^*p^* -C set which contradicting our assumption. Hence X is s^*p^* -connected.

Theorem 3.6: All s^*p^* connected space has connectedness.

Proof: Let X be a s^*p^* connected space. To show X is connected. If not, here, M and N are non-empty disjoint open sets. Hence $X = M \cup N$. We know each open set in X is known to be s^*p^* -O set in X . Thus $X = M \cup N$ where M and N are non-empty disjoint s^*p^* -O set in X . Therefore, this implies X is not s^*p^* connected. So, X is connected.

The below illustration demonstrates that the above theorems converse need not hold.

Example 3.7: Assume that $X = \{s, t\}$ $\tau = \{X, \phi\}$. Here, X is connected. s^*p^* -C sets are $\{X, \phi, \{s\}, \{t\}\}$. s^*p^* -O sets are $\{X, \phi, \{s\}, \{t\}\}$. Therefore, X is not s^*p^* -connected.

Theorem 3.8: Consider the TS X_1 . Let X_2 be a s^*p^* -connected subspace of X_1 . If X_1 can be expressed as the sum of two disjoint s^*p^* -O sets of X_1 then X_2 lies entirely within M or N .

Proof: Let $X_1 = M \cup N$ and $M \cap N = \phi$. Since M & N are s^*p^* -O set in X_1 then $M \cap X_2$ and $N \cap X_2$ are s^*p^* -O set in X_2 . Now $M \cap X_2$ and $N \cap X_2$ are disjoint and their sum is X_2 . They form a separation of X_2 if both were non empty, which is a contradiction. Hence one of them must be empty. Suppose $M \cap X_2 = \phi$. Then X_2 is equal to union of $(M \cap X_2)$ and $(N \cap X_2) \Rightarrow X_2 = \phi \cup (N \cap X_2) = N \cap X_2 \Rightarrow X_2 \subset N$. Similarly, if $N \cap X_2 = \phi$, then $X_2 \subset M$.

Theorem 3.9: Let s^*p^* -connected space of X be M . If $M \subset N \subset s^*p^*$ -cl(M). Prove that N is also s^*p^* -connected.

Proof: Assume that M is s^*p^* connected. Now we have to show that N is s^*p^* connected. If not, suppose that $N = C \cup D$. Since $M \subset N$, $M \subset (C \cup D)$, either $M \subset C$ or $M \subset D$. Suppose $M \subset C$, then s^*p^* -cl(M) $\subset s^*p^*$ -cl(C). Given that $N \subset s^*p^*$ -cl(M) $\Rightarrow N \subset s^*p^*$ -cl(M) $\subset s^*p^*$ -cl(C). Which, implies $N \subset s^*p^*$ -cl(C) then $N \cap D \subset s^*p^*$ -cl(C) $\cap D \Rightarrow N \cap D = \phi$. Now, $(C \cup D) \cap D = \phi$. This implies that $D = \phi$ which is a contradiction. Therefore, our assumption is not correct. Hence N is s^*p^* connected.

Theorem 3.10: Let $f: X_1 \rightarrow X_2$ be s^*p^* -continuous and surjective and is s^*p^* -connected thus X_2 is connected.

Proof: Consider X_2 is not connected. Suppose $X_2 = M \cup N$ where M and N are non-empty disjoint open sets in X_2 . If f is s^*p^* -continuous. Then inverse image of M and N are non-empty as well as disjoint

s^*p^* -O sets in X_1 . Given that X_1 is s^*p^* connected and f is onto. So, $X_1 = f^{-1}(M) \cup f^{-1}(N)$, which oppose the fact that X_1 is s^*p^* connected. Therefore, X_2 is connected.

Theorem 3.11: If $f: X_1 \rightarrow X_2$ is an onto, s^*p^* -irresolute and X_1 is s^*p^* connected. Show that X_2 is also s^*p^* connected.

Proof: Let us take X_2 is not s^*p^* connected. Let $X_2 = M \cup N$ Where M & N are disjoint non-empty s^*p^* -O sets in X_2 . Given that f is s^*p^* -irresolute, $f^{-1}(M)$ & $f^{-1}(N)$ are s^*p^* -O sets and since X_1 is s^*p^* -connected, $X_1 =$ union of $f^{-1}(M)$ and $f^{-1}(N) \Rightarrow X_1$ is not s^*p^* connected, which is a contradiction to our assumption. $\therefore X_2$ is s^*p^* -connected.

Theorem 3.12: The s^*p^* -cl of a s^*p^* connected space is s^*p^* connected.

Proof: H is a s^*p^* connected subset of (X, τ) . Prove that s^*p^* -cl(H) is s^*p^* connected. If not, s^*p^* -cl(H) can be written as s^*p^* -cl(H) = $M \cup N$, where M and N are disjoint s^*p^* -O sets. Now $H \subset s^*p^*$ -cl(H) = $M \cup N \Rightarrow H$ is either subset of M or N , suppose H is subset of M which implies s^*p^* -cl(H) $\subset s^*p^*$ -cl(M) this gives that s^*p^* -cl(H) $\cap N$ which is the subset of the intersection of s^*p^* -cl(M) & $N = \emptyset. \Rightarrow s^*p^*$ -cl(H) $\cap N = \emptyset$. Also, s^*p^* -cl(H) = $M \cup N \Rightarrow N \subset s^*p^*$ -cl(H) implies $N \cap s^*p^*$ -cl(H) = N . this implies $N = \emptyset$ which is a contradicts that N is nonempty. Similarly, if $H \subset N$ we will get $M = \emptyset$ which is a contradicts that M is non-empty. Therefore, s^*p^* -cl(H) has to be s^*p^* connected.

4. s^*p^* -COMPACT SPACE

Here, we determine the notions of s^*p^* compactness are defined and some of their properties are studied and various theorems will be discussed.

Definition 4.1: A Family $\{K_\alpha: \alpha \in J\}$ of s^*p^* -O sets in a TS X of a subset K_α is termed as a s^*p^* -O cover of X , if $\cup\{K_\alpha: \alpha \in J\} = X$.

Definition 4.2: The TS X is stated as s^*p^* compact if each s^*p^* -O cover $\{K_\alpha: \alpha \in J\}$ of X , that covers X as well as includes the finite subcollection.

Definition 4.3: Let TS be X . Let G be the subset of TS is term as s^*p^* compact if G is s^*p^* compact as a subspace of X .

Theorem 4.4: Each s^*p^* Compact space is also Compact.

Proof: Consider X represent the s^*p^* compact space. Let X have an open cover $\{K_\alpha: \alpha \in J\}$. Since each open set is s^*p^* -O set therefore $\{K_\alpha: \alpha \in J\}$ is a s^*p^* -O cover of X . Therefore, X is s^*p^* compact, the s^*p^* open cover $\{K_\alpha: \alpha \in J\}$ of X possess a sub cover which is finite say $\{K_\alpha: \alpha = 1, 2, \dots, n\}$ for X . So, X has become compact.

Definition 4.5: G be the subset of a TS X is stated that s^*p^* compact relative to X if each family $\{K_\alpha: \alpha \in J\}$ of s^*p^* -O subsets of X so that $G \subset \cup\{K_\alpha: \alpha \in J\}$ possess a finite subset J_0 of J then $G \subseteq \cup\{K_\alpha: \alpha \in J_0\}$.

Theorem 4.6: If $f: X_1 \rightarrow X_2$ be s^*p^* irresolute and a subset G of X_1 be s^*p^* compact relative to X_1 , following that $f(G)$ is s^*p^* compact relative to X_2 .

Proof: Consider $\{K_\alpha: \alpha \in J\}$ is the family of s^*p^* -O subsets of X_2 so that $f(G) \subset \cup\{K_\alpha: \alpha \in J\}$ then $G \subset \cup f^{-1}\{K_\alpha: \alpha \in J\}$ where $f^{-1}(K_\alpha)$ is s^*p^* -O in X_1 for each α . Given that f is s^*p^* irresolute $f^{-1}(K_\alpha)$ are s^*p^* -O in X_1 . But G is s^*p^* compact relative to X_1 . Thus, it possesses a finite subcover $G \subset \cup_{i=1}^n \{f^{-1}(K_i)\} \Rightarrow f(G) \subset \cup_{i=1}^n K_i \Rightarrow f(G)$ is s^*p^* compact.

Theorem 4.7: Under a s^*p^* Continuous map, the image of a s^*p^* compact space is also compact.

Proof: Assume that $f: X_1 \rightarrow X_2$ be s^*p^* continuous. Given that X_1 is a s^*p^* compact space on TS X_2 . Then the open cover of X_2 be $\{W_\alpha: \alpha \in J\}$. Given that f is s^*p^* continuous. After that $\{f^{-1}(W_\alpha): \alpha \in J\}$ is a s^*p^* open cover of X_1 . Because X_1 is s^*p^* compact. So, it forms a sub cover that is finite say $\{f^{-1}(W_1), f^{-1}(W_2), \dots, f^{-1}(W_n)\}$. Whenever f is onto $\{W_1, W_2, \dots, W_n\}$ is an open cover of X_2 , that one is finite. Hence X_2 is compact.

Theorem 4.8: If the function $f: X_1 \rightarrow X_2$ is a s^*p^* irresolute and a bijective map and X_1 is s^*p^* compact then X_2 also s^*p^* compact space.

Proof: Consider that $\{H_\alpha: \alpha \in J\}$ be a s^*p^* open cover of X_2 . Hence, $X_2 = \cup H_\alpha$. Because f is a bijection, we have $f(X_1) = X_2$ or $f^{-1}(X_2) = X_1$. But $f^{-1}(\cup H_\alpha) = \cup\{f^{-1}(H_\alpha)\}$. Given that f is s^*p^* irresolute $f^{-1}(H_\alpha)$ is s^*p^* -O set in X_1 for each $\alpha \in J$. But X_1 is s^*p^* compact. Therefore, there exists finitely many indices $\alpha_1, \alpha_2, \dots, \alpha_n$ such that $X = \cup_{i=1}^n f^{-1}(H_{\alpha_i})$. Therefore, $X_2 = f(\cup (f^{-1}(H_{\alpha_i}))) = \cup H_{\alpha_i}$ where $i = 1, 2, 3, \dots, n \Rightarrow X_2$ is s^*p^* compact.

Theorem 4.9: Every s^*p^* -C subset of a s^*p^* compact space is s^*p^* compact relative to X .

Proof: Assume s^*p^* -C subset of X be A . Then A^c is s^*p^* -O set. Let $\rho = \{H_\alpha: \alpha \in J\}$ be a s^*p^* -O cover of A . After that $M = \rho \cup A^c$ is an s^*p^* -O cover of X . Thus, $X \subset \bigcap \{H_\alpha: \alpha \in J\} \cup A^c$. since X is s^*p^* compact \exists a finite subcover from M say $(H_1 \cup H_2 \cup H_3 \dots H_n) \cup A^c$ for X . But A and A^c are disjoint and hence $A \subset H_1 \cup H_2 \cup H_3 \dots H_n$. The open cover ρ possesses a finite subcover. So, A is s^*p^* compact.

Theorem 4.10: A TS X is s^*p^* compact iff each family of s^*p^* -C sets of X with the Finite intersection property contains not empty intersection.

Proof: Consider X be s^*p^* compact. Then take $\{H_\alpha: \alpha \in J\}$ be an arbitrary family of s^*p^* -C sets of X with Finite intersection property such that $\bigcap \{H_{\alpha_i}: i = 1, 2, \dots, n\} \neq \emptyset$. Now showing that $\bigcap \{H_\alpha: \alpha \in J\} \neq \emptyset$. Suppose this is not true. Then we have $\bigcap \{H_\alpha: \alpha \in J\} = \emptyset$. By taking complement, we have $\bigcup \{H_\alpha^c: \alpha \in J\} = X$. since each H_α is s^*p^* closed set, H_α^c is open. Therefore $\{H_\alpha^c: \alpha \in J\}$ becomes an s^*p^* open cover for X . But X is s^*p^* compact. Therefore, it exists finitely many indices $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$. such that $X = \bigcup \{H_{\alpha_i}^c: i = 1, 2, \dots, n\}$. taking complement both sides we get $\bigcap \{H_{\alpha_i}: i = 1, 2, \dots, n\} = \emptyset$ which contradicts our assumption. So, $\bigcap \{H_\alpha: \alpha \in J\} \neq \emptyset$. Conversely let each family of s^*p^* -C sets with Finite intersection property has a non-empty intersection. Consider $B = \{G_\alpha: \alpha \in J\}$ then B is a s^*p^* open cover of X and $X = \bigcup \{G_\alpha: \alpha \in J\}$. Using complement, we get $\emptyset = \bigcap \{G_\alpha^c: \alpha \in J\}$. but G_α^c is s^*p^* -C set. Thus, the collection of s^*p^* -C subsets has empty intersection. Hence, it does not satisfy Finite intersection property. Hence \exists a finite number of s^*p^* closed set $H_{\alpha_i}^c$ where $i = 1, 2, \dots, n$ with non-empty intersection. That is $\bigcap \{H_{\alpha_i}^c: i = 1, 2, \dots, n\} = \emptyset$. Again taking complement $\bigcup \{H_{\alpha_i}^c: i = 1, 2, \dots, n\} = X$. therefore B contains a subcover for X . Which is finite. Hence X is s^*p^* compact.

5. Conclusion

In this study, we investigate different type of connected and compact spaces which is called as s^*p^* connected and s^*p^* compact spaces in topological spaces using s^*p^* open set. Some of characterizations and their fundamental properties are studied. Also examined the relationship between newly founded spaces with the existing connected and compact spaces.

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