Strongly Isolate Perfect Domination in Graphs

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Abstract: A dominating set D^* of a graph G^+ is said to be an isolate dominating set (IDS) if $< D^* >$ has at least one isolated vertex. The ID number of G^+ is represented by $\gamma_0(G^+)$. An ID-set D^* is considered as strongly isolate dominating set (SIDS) if there exists a $\in D^*$ such that $N_2(a) \cap D^* = \phi$, where $N_2(a) = \{a: d(a,b) \leq 2 \text{ and } a \neq b\}$. A dominating set D^* is called a perfect dominating set if every vertex in V (G) $- D^*$ has exactly one neighbour in D^*. By using the above concept and the definition of SID, we define a new concept called "Strongly Isolate Perfect Domination"(SIPD). An isolate dominating set D^* is said to be strongly isolate perfect dominating set if there exists $a \in D$ such that $N_2(a) \cap D^* = \phi$ and D^* is a perfect dominating set of G^+. This paper involves some basic features of SIPDS and compare SIPDS with dominating set, ID-set and efficient dominating set (EDS). At the end, includes SIPD number of path, cycle, complete bi- partite graph, complete b- partite graph and some group of graphs.

Keywords: Domination, isolate domination, strongly isolate perfect domination.

1. Introduction

The general idea of ID in graphs" has been exposed by I.Shahul Hamid, S.Balamurugan [1], in 2016. A dominating set D^* of a graph G^+ is considered as an isolate dominating set of G^+ if $< D^* >$ has a minimum of one isolated vertex. The minimum cardinality of an ID- set of G^+ is known as ID number. The ID number of a graph G^+ is denoted by $\gamma_0(G^+)$. An ID- set of cardinality $\gamma_0(G^+)$ is called γ_0 - set. Through the application of the ID concept, we established a new domination criterion, specifically "Strongly Isolate Domination"(SID). An ID set D^* is called SID- set if there exists $v \in D^*$ such that $N_2(a) \cap D^* = \phi$, where $N_2(a) = \{a: d(a,b) \leq 2 \text{ and } a \neq b\}$. The minimum cardinality of a SID-set of G^* are known as the SID number. The SID number is represented by $\gamma_0{}^s(G^+)$. A SID set of cardinality $\gamma_0{}^s(G^+)$ is known as $\gamma_0{}^s$- set.
A dominating set D^* is called a perfect dominating set if every vertex in $V(G^+) - D^*$ has exactly one neighbour in D^*. By using the above concept and the definition of SID, we define a new concept called "Strongly Isolate Perfect Domination"(SIPD). An isolate dominating set D^* is said to be strongly isolate perfect dominating set if there exists $a \in D$ such that $N_2(a) \cap D^* = \phi$ and D^* is a perfect dominating set of G^+.
An efficient dominating set (EDS) $D^* \subseteq V$ in a graph $G^+ = (V, E)$ is a dominating set with extra property that the closed neighbourhood N[x] of every vertex $x \in V$ contains only one vertex in D^*. Let x is a point on a connected graph G^+. The eccentricity $e(x)$ of x is defined by $e(x) = \max\{d(x,y)/x \in V(G^+)\}$. This paper explores certain properties of SIPD and the SIPD number in various groups of graphs.

2. Basics of strongly isolate perfect dominating set.

This section focuses on deriving certain bounds for the SIPD number of connected graphs.
Theorem 2.1: For any graph G^+, $\max\{\gamma(G^+), \gamma_0(G^+), \gamma_p(G^+), \gamma_0{}^s(G^+)\} \leq \gamma_{0,p}{}^s(G^+)$.
Proof 1: Since each and every SIPDS of G^+ is also a dominating set of G^+, we have $\gamma(G^+) \leq \gamma_{0,p}{}^s(G^+)$.

Since each and every SIPDS of G^+ is also an isolate dominating set of G^+, we have $\gamma_0(G^+) \leq \gamma_{0,p}{}^s(G^+)$. Since each and every SIPDS of G^+ is also a perfect dominating set of G^+, we have $\gamma_p(G^+) \leq \gamma_{0,p}{}^s(G^+)$. Since each and every SIPDS of G^+ is also a strongly isolate dominating set of G^+, we have $\gamma_0{}^s(G) \leq \gamma_{0,p}{}^s(G^+)$.

The next theorem gives the SIPD number of disconnected graphs which admit SIPD-sets.

Theorem 2.3: Let G^+ be a disconnected graph with p components $F_1, F_2, \dots \dots, F_p$ such that the first q components $F_1, F_2, \dots \dots, F_q$ admit SIPDS. Then $\gamma_{0,p}{}^s(G^+) = \min_{1\leq m\leq q}\{b_m\}$, where $b_m = \gamma_{0,p}{}^s(F_m) + \sum_{1\leq s\leq p,\ s\neq m}\{\gamma_p(F_s)\}$for $1 \leq m \leq q$.

Proof 3: Let us assume $b_1 = \min_{1\leq m\leq q}\{b_m\}$. Let T^+ be a $\gamma_{0,p}{}^s$ - set of F_1 and let ${X_m}^+$ be a γ_p-set of F_m for every m with $2 \leq m \leq p$. Then the set $T^+ \cup (\bigcup_{m=2}^{p} {X_m}^+)$ is a SIPD- set of G^+. Therefore, $\gamma_{0,p}{}^s(G^+) \leq \gamma_{0,p}{}^s(F_1) + \sum_{m=2}^{p} \gamma_p\gamma(F_m) = b_1$. Now consider T' be a minimal SIPD set of G^+. Hence the set T' should intersect the vertex set $V(F_m)$ for each m with $1 \leq m \leq p$. In addition, there exists r such that $T' \cap V(F_r)$ is a minimal SIPDS of F_r and $1 \leq r \leq q$. Furthermore, every $1 \leq m \leq p$, $m \neq r$, the set $T' \cap V(F_m)$ is a perfect dominating of F_m. Thus $|T'| \geq \gamma_{0,p}{}^s(F_r) + \sum_{1\leq r\leq p, s\neq m} \gamma_p(F_m) \geq b_1$ and so $\gamma_{0,p}{}^s(G^+) = b_1$.

Remark 2.1 If the eccentricity of a vertex p of a graph G^+ is ≤ 2, then the vertex is not SI and such vertex cannot be part of any SIPDS of G^+.

Theorem 2.4: Suppose G^+ be an ordered connected graph with $n \geq 4$. Then $\gamma_{0,p}{}^s(G^+) = 2$ if and only if there are two vertices $c, d \in D^* \subseteq V(G^+)$ therefore, $N(c) \cap N(d) = \phi$ and $N[c] \cup N[d] = V(G^+)$.

Proof 4 : Suppose $\gamma_{0,p}{}^s(G^+) = 2$. Then there exists a SIPD-set $D^* = \{c, d\}$ and c is strongly isolated in $< D^* >$. Suppose there exists $w \in N(c) \cap N(d)$. Then $d \in N_2(c)$, a contradiction to c is strongly isolated. Thus $N(c) \cap N(d) = \phi$. By the definition of dominating set $N[D^*] = V(G^+)$. That is, $N[c] \cup N[d] = V(G^+)$. Conversely assume that $N(c) \cap N(d) = \phi$ and $N[c] \cup N[d] = V(G^+)$. Clearly, $D^* = \{c, d\}$ is a SIPD- set of G^+. Therefore, $\gamma_{0,p}{}^s(G^+) = 2$.

Theorem 2.5 A tree with $\dim(T^*) = 3$ admits strongly isolate perfect dominating set if and only if T^* is a Mob graph.

Proof 5: Let G^+ be a connected graph. Assume that G^+ admits SIPD set, say D^* and let z be a strongly isolated vertex in $< D^* >$. Figure 2.1 illustrate the longest path $P : u_1 - u_2 - u_3 - u_4$ in G^+, since $\dim(G^+) = 3$.

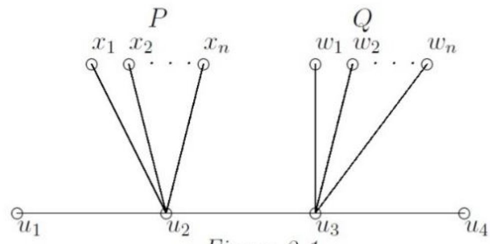

Figure 2.1

Let P represent the collection of vertices nearest to u_2 except u_1 and u_3, Q represent the collection of vertices nearest to u_3 except u_2 and u_4. In Remark 2.1, $u_2, u_3 \notin D^*$. Suppose $= u_1$, then by the definition of SIPDS, u_2, $N(u_2) \notin D^*$. Similarly, we can get contradiction when $= u_4$.

Assume that P and Q are not empty.

Case 1: Suppose $z \in P$.

In any case imagine that $z = x_1$, then $u_2, u_1 \notin D^*$ and hence D^* cannot dominate u_1, a contradiction. Similarly, $z \neq x_i$ for all $1 \leq i \leq |P|$.

Case 2: Suppose $z \in Q$.

In any case imagine that $= w_1$, then $u_3, u_4 \notin D^*$ and hence D^* cannot dominate u_4, a contradiction. Similarly, $z \neq w_i$ for all $1 \leq i \leq |Q|$. Thus either P or Q must be empty. Thus G^+ is a Mob graph.

Converse part is trivial.

Theorem 2.6 : Let G^+ be a graph such that $\dim(G^+) \leq 2$ also $\Delta(G^+) < n - 1$, then G^+ does not admit SIPD-set.

Proof 6 : On the contrary, suppose G^+ admits SIPDS, let D be a SIPDS. Let u be a strongly isolated vertex in $< D >$. Since $\deg G(u) < n - 1$, there are some un dominated vertices in the graph $< G -$

$N[u] >$. Thus D must have one more element $v \neq u$. Since $diam(G^+) \leq 2$ and u is strongly isolated in < D >, $N_2(u) \cap D = \phi$. But $N_2(u)$ contains a vertex v and D could not be perfect, a contradiction.

3. Comparative study of strongly isolate perfect dominating set.

In this section, we compare SIPDS with EDS set, dominating set and IDS.
Theorem 3.1 Let D^* be a SIPDS in which each vertex of D^* is strongly isolated. Then D^* is an efficient dominating set.
Proof 7: Let D^* be a SIPDS of a graph G in which each vertex is strongly isolated. Claim, D^* is an EDS.
Case 1: Let $v \in V - D^*$. Suppose there are two vertices $c, f \in D^* (c \neq f)$ such that v is adjacent with both c and f. Since c is strongly isolated, $N_2(c) \cap D^* = \phi$. But $N_2(c) \cap D^*$ contains a vertex f, a contradiction. Since D^* is a SIPDS, v is adjacent with only one vertex of D^*. Thus $|N[v] \cap D^*| = 1$. So D^* is an EDS.
Theorem 3.2 : Every dominating set is an efficient dominating set if and only if it is a strongly isolate perfect dominating set.
Proof 8: Let G be a graph and D^* be EDS of G. By the definition of EDS, each vertex of D^* is perfect. To prove D^* is a SIDS, it is enough to prove that D^* has a strongly isolated vertex. Let $a \in D^*$. Suppose a is not strongly isolated, $N_2(a) \cap D^* \neq \phi$. Let $x \in N_2(a) \cap D^*$. Since D^* is independent, there exists $y \in V - D^*$ such that $(a, y), (y, x) \in E(G)$. Thus y is dominated twice by the vertex of D^*, namely, a and x, a contradiction. Thus a is strongly isolated and so D^* is SIPDS.
Converse part is obvious.
Theorem 3.3: For any integer $n \geq 1$, there exist a graph G^+ such that $\gamma(G^+) = \gamma_0(G^+) = \gamma_{0,p}{}^s (G^+) = n$.
Proof 9: Consider a path P_n with $V(P_n) = \{v_1, v_2 \ldots \ldots, v_n\}$. For each v_i $(1 \leq i \leq n)$, attach a pendent vertex u_i $(1 \leq i \leq n)$ and let G^+ be the resultant graph. Note that $n = \gamma(G^+) = \gamma_{0,p}$ (G^+). Also $\{u_1, u_2 \ldots \ldots, u_n\}$ is a SIPDS with n elements and so $\gamma_{0,p}{}^s (G^+) \leq n$. By Theorem 2.1, $\gamma(G^+) = \gamma_0(G^+) = \gamma_{0,p}{}^s (G^+) = n$.
Theorem 3.4 : Let $\boldsymbol{p}$ and $\boldsymbol{q}$ are two integers such that $\boldsymbol{p} < \boldsymbol{q}$. Then there exist a graph $\boldsymbol{G^+}$ such that $\gamma(\boldsymbol{G^+}) = \boldsymbol{p}$ and $\boldsymbol{\gamma_{0,p}{}^s} (\boldsymbol{G^+}) = \boldsymbol{q}$.
Proof 10: Consider the complete graph $\boldsymbol{K_P}$ of order $\boldsymbol{p}$ with $V(\boldsymbol{K_p}) = \{\boldsymbol{v_1}, \boldsymbol{v_2} \ldots \ldots, \boldsymbol{v_P}\}$.
For each $\boldsymbol{v_i}$ $(\boldsymbol{2} \leq \boldsymbol{i} \leq \boldsymbol{p})$, attach a pendent vertex $\boldsymbol{u_i}$ $(\boldsymbol{2} \leq \boldsymbol{i} \leq \boldsymbol{p})$. Let W be any graph which admits SIPDS with $\boldsymbol{\gamma}(W) = \boldsymbol{\gamma_{0,p}{}^s}(W) = \boldsymbol{q - p + 1}$ and join each and every vertex of W to $\boldsymbol{v_1}$ and $\boldsymbol{G^+}$ be the resultant graph. Let $\boldsymbol{D^*}$ be a dominating set of $\boldsymbol{G^+}$. To dominate the vertices of V (W) and $\boldsymbol{v_1}$, $\boldsymbol{D^*}$ should take in either $\boldsymbol{v_1}$ or minimum of one vertex of V (W). For each $\boldsymbol{i = 2}$ to $\boldsymbol{a}$, to dominate the vertices $\boldsymbol{v_i}$ and $\boldsymbol{u_i}$, $\boldsymbol{D^*}$ should take in either $\boldsymbol{u_i}$ or vertex $\boldsymbol{v_i}$ and so $\gamma(\boldsymbol{G^+}) \geq p$. Since $V(\boldsymbol{K_p})$ is a dominating set of $\boldsymbol{G^+}$, we have $\gamma(\boldsymbol{G^+}) \leq \boldsymbol{p}$ and so $\gamma(\boldsymbol{G^+}) = \boldsymbol{p}$.
Let $\boldsymbol{D^*}$ be a SIPDS $\boldsymbol{G^+}$ and $\boldsymbol{u}$ be a strongly isolated vertex in < $\boldsymbol{D^*}$ >. Since $e(\boldsymbol{v_i}) = \boldsymbol{2}$ for all $\boldsymbol{1} \leq \boldsymbol{i} \leq \boldsymbol{p}, \boldsymbol{v_i} \notin \boldsymbol{u}$.
Case 1: Assume $u = \boldsymbol{u_i}$ for any $\boldsymbol{2} \leq \boldsymbol{i} \leq \boldsymbol{p}$. Thus $V(\boldsymbol{K_p}) \cap \boldsymbol{D^*} = \phi$ and so $\boldsymbol{u_i} \in \boldsymbol{D^*}$ for all $\boldsymbol{2} \leq \boldsymbol{i} \leq \boldsymbol{p}$. Since $\boldsymbol{v_1} \notin \boldsymbol{D^*}$, to dominate each and every of W , $\boldsymbol{D^*}$ must take in $\boldsymbol{q - p + 1}$ vertices(since $\boldsymbol{\gamma_{0,p}{}^s}$ $(\boldsymbol{W}) = \boldsymbol{q - p + 1}$). Thus $\boldsymbol{\gamma_{0,p}{}^s} (\boldsymbol{G^+}) \geq (\boldsymbol{p - 1}) + (\boldsymbol{q - p + 1}) = \boldsymbol{q}$ and so $\boldsymbol{\gamma_{0,p}{}^s} (\boldsymbol{G^+}) \geq q$.
Case 2: Assume $\boldsymbol{u} \in V(W)$. Thus $V(\boldsymbol{K_p}) \cap \boldsymbol{D^*} = \phi$ and so $\boldsymbol{u_i} \in \boldsymbol{D^*}$ for all $\boldsymbol{2} \leq \boldsymbol{i} \leq \boldsymbol{p}$. Since $\boldsymbol{v_1} \notin \boldsymbol{D^*}$, to dominate each and every vertex of W , $\boldsymbol{D^*}$ should take in $\boldsymbol{q - p + 1}$ vertices(since $\boldsymbol{\gamma_{0,p}{}^s}(W) = \boldsymbol{q - p + 1}$). Thus $\boldsymbol{\gamma_{0,p}{}^s} (\boldsymbol{G^+}) \geq (\boldsymbol{p - 1}) + (\boldsymbol{q - p + 1}) = \boldsymbol{q}$. Thus, $\boldsymbol{\gamma_{0,p}{}^s} (\boldsymbol{G^+}) \geq \boldsymbol{q}$. From Case 1 and Case 2, $\boldsymbol{\gamma_{0,p}{}^s} (\boldsymbol{G^+}) \geq \boldsymbol{q}$. Let D' be a $\boldsymbol{\gamma_{0,p}{}^s}$ - set of W. Also $D' \cup \{\boldsymbol{u_2}, \boldsymbol{u_3} \ldots \ldots, \boldsymbol{u_p}\}$ remains SIPDS with $(\boldsymbol{p - 1}) + (\boldsymbol{q - p + 1}) = \boldsymbol{q}$ - elements. Thus $\boldsymbol{\gamma_{0,p}{}^s} (\boldsymbol{G^+}) \leq \boldsymbol{q}$ and so $\boldsymbol{\gamma_{0,p}{}^s} (\boldsymbol{G^+}) = \boldsymbol{q}$.
4 Strongly isolate perfect domination number of certain graphs.
This section, we obtain the SIPD number for paths, cycles and various group of graphs.
Theorem 4.1 : For a path $\boldsymbol{P_m}$ of order $(\boldsymbol{m \geq 4})$, we have $\boldsymbol{\gamma_{0,p}{}^s} (\boldsymbol{P_m}) = \lceil \boldsymbol{m/3} \rceil$.
Proof 12: Let $\boldsymbol{V}(\boldsymbol{P_m}) = \{\boldsymbol{w_1}, \boldsymbol{w_2} \ldots \ldots, \boldsymbol{w_m}\}$ and $\boldsymbol{D}$ be a SIPDS of $\boldsymbol{P_m}$. By Theorem 2.1, $\lceil \boldsymbol{m/3} \rceil = \boldsymbol{\gamma}(\boldsymbol{P_m}) \leq \boldsymbol{\gamma_{0,p}{}^s} (\boldsymbol{P_m})$.
Case 1: When $\boldsymbol{m = 4}$ or $\boldsymbol{m = 5}$.
Since the set $\boldsymbol{D} = \{\boldsymbol{w_1}, \boldsymbol{w_4}\}$ is a SIPDS of $\boldsymbol{P_m}$ with 2 elements. Thus $\boldsymbol{\gamma_{0,p}{}^s} (\boldsymbol{P_m}) \leq \lceil \boldsymbol{m/3} \rceil$.
Case 2: When $\boldsymbol{m \geq 6}$.

Sub case 1: Suppose $m = 3r$ for $r \geq 2$.
Since the set $D = \{w_{3i-1} : 1 \leq i \leq r\}$ SIPDS of P_m with r - elements. Thus $\gamma_{0,p}{}^{s}(P_m) \leq r = \lceil m/3 \rceil$.
Sub case 2: Suppose $m = 3r + 1$ or $m = 3r + 2$ for $r \geq 2$.
Since the set $D = \{w_{3i-1} : 1 \leq i \leq r\}$ is a SIPDS of P_m with $r + 1$- elements. Thus $\gamma_{0,p}{}^{s}(P_m) \leq + 1 = \lceil m/3 \rceil$.
Theorem 4.2 : For a cycle C_m of order ($m \geq 6$), we have $\gamma_{0,p}{}^{s}(C_m) = \lceil m/3 \rceil$+1, if $n \equiv 2mod(3)$; $\left\lceil \frac{m}{3} \right\rceil$, otherwise.
Proof 13: Let $V(C_m) = \{z_1, z_2, \ldots\ldots, z_m\}$ and D be a SIPDS of C_m. By Theorem 2.1, $\lceil m/3 \rceil = \gamma(C_m) \leq \gamma_{0,p}{}^{s}(C_m)$.
When m ≥ 6
Case 1: Suppose m = 3r for r ≥ 2.
Since the set $D = \{z_{3i-1} : 1 \leq i \leq r\}$ is a SIPDS of C_m with r -$\lceil m/3 \rceil$ elements.
Case 2: Suppose $m = 3r + 1$ for $r \geq 2$.
Since the set $D = \{z_{3i-2} : 1 \leq i \leq r\} \cup \{z_m\}$ is a SIPDS of C_m with $r = \lceil m/3 \rceil$- elements.
Case 3: Suppose $m = 3r + 1$ for $r \geq 2$.
Since the set $D = \{z_{3i-2} : 1 \leq i \leq r\} \cup \{z_m\}$ is a SIPDS of C_m with $r = \left\lceil \frac{m}{3} \right\rceil + 1$- elements.
By Theorem 2.1, $\lceil m/3 \rceil = \gamma(C_m) \leq \gamma_{0,p}{}^{s}(C_m)$ and so $\gamma_{0,p}{}^{s}(C_m) = \lceil m/3 \rceil$+1, if $n \equiv 2mod(3)$; $\left\lceil \frac{m}{3} \right\rceil$, otherwise.
Remark 4.1: For the graph, $\{v_1\}$ is a SIPDS and $\gamma_{0,p}{}^{s}(C_3) = 1$.

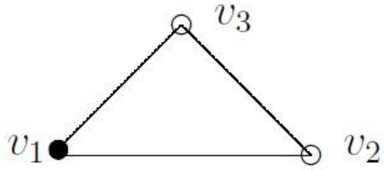

Figure 4.1: C_3

Remark 4.2: The graph $C_n(n = 4, 5)$ does not admit SIPDS.
Proof 14: Since $e(v) = 2$ for any vertex v of C_4 and C_5 , C_4 and C_5 does not admit SIPDS.
Theorem 4.3: Let G^+ be an order m ≥ 1 graph. Then $\gamma_{0,p}{}^{s}(G^+) = 1$ if and only if $\Delta(G) = m - 1$.
Proof 15: Assume $\gamma_{0,p}{}^{s}(G^+) = 1$ and let F = {z} be a minimum SIPDS of G^+. If G^+ is a singleton graph, after that we are finished. Suppose G^+ is not singleton graph. Obviously, each vertex in $< V(G^+) - F >$ is adjacent to z. Since |V (G^+)| = m, $degG(z) = m - 1$. Consequently, $\Delta(G^+) = m - 1$. The converse is true.
The next corollary follows straight from the above theorem.
Corollary 4.1 Let S_{n-1}, K_n and W_{n-1} be star graph, complete graph and wheel graph of n ≥ 2 vertices, sequentially. Then $\gamma_{0,p}{}^{s}(S_{n-1}) = \gamma_{0,p}{}^{s}(K_n) = \gamma_{0,p}{}^{s}(W_{n-1}) = 1$.
Theorem 4.4: The complete bipartite graph $K_{p,q} = (P, Q)$ admits SIPDS if and only if $p = 1$ or $q = 1$.
Proof 16: Given $K_{p,q}$ = (P, Q) be a complete bipartite graph. Let P = $\{r_i : 1 \leq i \leq p\}$ and $Q = \{s_i : 1 \leq i \leq q\}$. Take D^* be a SIPDS and x be the SI vertex in < D^* >. Assume $K_{p,q}$ admits strongly isolate perfect dominating set. Claim, $p = 1$ or $q = 1$. Assume $p \neq 1$ and $q \neq 1$. In any case assume that x ∈ P. As x is isolated in < D^* >, D^* cannot have any vertex of Q. Consequently to dominate every vertex of P, D^* should necessarily add each and every vertices of P. Hence P = D^*. Here $N_2(x) \cap D^* \neq \phi$, a contradiction. Similarly, we get the same contradiction, when $x \in Q$. Therefore, $p = 1$ or $q = 1$.
Converse part is trivial.
Theorem 4.5: Let $b \geq 2$ be an integer and $G = K_{p_1, p_2, \ldots\ldots, p_b} = (P_1, P_2, \ldots\ldots, P_b)$ be a complete b-partite graph. If $\mathbf{p_j}$ = 1 for an integer $\mathbf{j}$ with $1 \leq \mathbf{j} \leq \mathbf{b}$ if and only if G admits SIPDS.

Proof 17 : Given G = $K_{p_1,p_2,.....,p_b}$ = $(P_1, P_2, ... , P_b)$ be a complete b-partite graph. Take a SIPDS G and a SI vertex y. Claim, p_j = 1 for an integer j with $1 \le j \le b$. In any case, we may suppose that $y \in P_1$. As y is isolated in < D >, $D \cap (P_2 \cup P_3 \cup . . . \cup P_b) = \phi$. Thus to dominate each and every vertex of P_1, D should take all the vertices of P_1. Thus D = P_1. Here $N_2(y) \cap D \neq \phi$, a contradiction.
Converse part is trivial.
Theorem 4.6 : For any Helm graph H_m with $2m + 1$ vertices, we have $\gamma_{0,p}{}^{s}(H_m) = m$.
Proof 18 :Let H_m be a Helm graph. Let V (H_m) = $\{q_i : 1 \le i \le m\} \cup \{p_i : 1 \le i \le m\} \cup \{x\}$ and $E(H_m) = \{q_i p_i, xq_i : 1 \le i \le m\} \cup \{q_i q_{i+1} : 1 \le i \le m - 1\} \cup \{q_1 q_m$. By Theorem 2.1, $m \le \gamma(H_m)$ $\le \gamma_{0,p}{}^{s}(H_m)$. Since the set D = $\{p_i : 1 \le i \le m - 1\} \cup \{p_m\}$ is a SIPDS of H_m with m elements, $\gamma_{0,p}{}^{s}(H_m) \le m$.
Corollary 4.2: The flower graph does not admit SIPDS.
Theorem 4.7: For every Comb graph $P_n * k_1$ of order n($\ge$ 3), $\gamma_{0,p}{}^{s}(P_n * k_1)$ = n.
Proof 19: Let $P_n * k_1$ be a Comb graph and let V $(P_n * k_1) = C \cup D$, where V $(C) = \{c_i : 1 \le i \le n\}$, V $(D) = \{d_i : 1 \le i \le n\}$ and $E(P_n * k_1) = \{c_i c_{i+1} : 1 \le i \le (n - 1)\} \cup \{c_i d_i : 1 \le i \le n\}$. Since each d_i is pendent, either d_i or c_i must be in every dominating set and so $\gamma(G) \ge n$. Consequently by Theorem 2.1, $n \le \gamma_{0,p}{}^{s}(P_n * k_1)$. Since the set D = $\{d_i : 1 \le i \le n\}$ is a SIPDS of $P_n * k_1$ and so $\gamma_{0,p}{}^{s}(P_n * k_1) \le n$.
Theorem 4.8: For every graph G^+ of order m ($\ge$ 1), $\gamma_{0,p}{}^{s}(G^+ * k_1)$ = m.
Proof 20: Let V $(G^+) = \{u_1, u_2 ......, u_m\}$ for each $1 \le i \le m$. Let v_i be the pendent vertex such that $u_i v_i \in E(G^+ * k_1)$ and D be a SIPDS of $G^+ * k_1$. To dominate the vertex v_i, either u_i or v_i must be in D. Thus $|D| \ge m$ and so $\gamma_{0,p}{}^{s}(G^+ * k_1) \ge m$. Since the set D = $\{v_1, v_2 ......, v_m\}$ is a SIPDS of $G^+ * k_1$. Thus $\gamma_{0,p}{}^{s}(G^+ * k_1) \le m$.
Theorem 4.9: For any firecracker graph $F(s, t)$ of order $st(s, t \ge 2)$, we have $\gamma_{0,p}{}^{s}(F(s,)t) = s$.
Proof 21: By Theorem 2.1, $= \gamma(F(s,t)) \le \gamma_{0,p}{}^{s}(F(s,t))$. Let V (F(s, t) = $\{u_{k,r} : k = 1, 2, 3 \ldots, t;\ r = 1, 2, 3 \ldots, s\}$ and let $E(F(s,t) = \{u_{1,r}\, u_{k,r}$: k = 2, 3 . . . t; r = 1, 2, 3 . . . , s} ∪ $\{u_{b,r}\, u_{b,r+1}$: k = 1, 2, 3 . . . , (s − 1)}. Since the set D = $\{u_{1,r}$: r = 1, 2, 3 . . . , s} is a SIPDS of $F(s,t)$ and so $\gamma_{0,p}{}^{s}(F(s,t)) \le s$.
Theorem 4.10: The sun graph sun(s) with 2s vertices, admits SIPDS with $\gamma_{0,p}{}^{s}$ (sun(s)) = s.
Proof 22: By Theorem 2.1, s = γ(sun(s)) $\le \gamma_{0,p}{}^{s}$ (sun(s)). Let V (sun(s)) = $\{p_i : 1 \le i \le s\} \cup \{q_i : 1 \le i \le s\}$ and $E(sun(s)) = \{p_i q_i : 1 \le i \le s\}$. Let D be a SIPDS of sun(s). Since the set D $= \{p_i : 1 \le i \le s\}$ is a SIPDS of sun(s) with s elements so $\gamma_{0,p}{}^{s}(sun(s)) \le s$.
Theorem 4.12: The book graph B_n, $n \ge 2$ admits strongly isolate perfect dominating set and $\gamma_{0,p}{}^{s}(B_n)$ = n.
Proof 24: Let f and d be the centre vertices of two copies of S_{n+1} in the Cartesian product $S_{n+1} \circ P_2$. Let $f_1, f_2, ... , f_n$ be the vertices adjacent to f and $d_1, d_2, ... , d_n$ be the vertices adjacent to d. By Remark 2.1, f and d are not in SIPDS of B_n. Thus for each $1 \le i^* \le n$, to dominate the vertex f_{i^*}, each f_{i^*} or d_{i^*} must be in any SIPDS of B_n and so $\gamma_{0,p}{}^{s}(B_n) \ge n$. Since the set $\{f_1\} \cup \{f_2, f_3, ... , f_n\}$ remains a SIPDS with n vertices, $\gamma_{0,p}{}^{s}(B_n) \le n$.

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